Confidence bands for densities, logarithmic point of view

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Abstract

Let f be a probability density and C be an interval on which f is bounded away from zero. By establishing the limiting distribution of the uniform error of the kernel estimates f_n of f, Bickel and Rosenblatt (1973) provide confidence bands B_n for f on C with asymptotic level $1-\alpha \in]0,1[$. Each of the confidence intervals whose union gives B_n has an asymptotic level equal to one; pointwise moderate deviations principles allow to prove that all these intervals share the same logarithmic asymptotic level. Now, as soon as both pointwise and uniform moderate deviations principles for f_n exist, they share the same asymptotics. Taking this observation as a starting point, we present a new approach for the construction of confidence bands for f, based on the use of moderate deviations principles. The advantages of this approach are the following: (i) it enables to construct confidence bands, which have the same width (or even a smaller width) as the confidence bands provided by Bickel and Rosenblatt (1973), but which have a better aymptotic level; (ii) any confidence band constructed in that way shares the same logarithmic asymptotic level as all the confidence intervals, which make up this confidence band; (iii) it allows to deal with all the dimensions in the same way; (iv) it enables to sort out the problem of providing confidence bands for f on compact sets on which f vanishes (or on all $\mathbb{R}^d$), by introducing a truncating operation.

Key words and phrases: Density; Kernel estimator; Asymptotic logarithmic level; Asymptotic almost sure confidence regions; Moderate deviations principles.

1   Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed $\mathbb{R}^d$-valued random variables with bounded probability density function f. The problem of computing confidence bands for f is of central interest in nonparametric statistics.

One main known approach to the construction of confidence bands for f is due to Bickel and Rosenblatt (1973) in the case $d = 1$ and to Rosenblatt (1976) in the case $d \geq 2$, and is based on the limiting distribution of the normalized uniform error of the kernel density estimate. The approach of Bickel and Rosenblatt has been extended, in the unidimensional case, in several directions; among others, let us cite Mack (1982) and Liu and Ryzin (1986) for the extension to other types of density estimates, Burke and Horvath (1984) and Mielniczuk (1987) for the censored data case, Xu and Martinsek (1995), Martinsek and Xu (1996), Sun and Zhou (1998) for the construction of sequential confidence bands, and Giné, Koltchinskii and Sakhanenko (2003, 2004) for different asymptotics of the weighted uniform error. Another common technique for constructing confidence bands is through the bootstrap, which is in particular used for bias estimation; see, for example, Hall (1992) for the density and Härdle and Marron (1991) for the regression. For other approaches, see Hall and Titterington (1988) and Hall and Owen (1993).

Our object in this paper is to present a new approach, based on the use of Moderate Deviations Principles (MDP) of the normalized uniform error of the kernel density estimates. We avoid bias estimation by a slight undersmoothing, which is shown in Hall (1992) to be more efficient than explicit bias correction when the goal is to minimize the coverage error of the confidence band.

The use of large and moderate deviations in statistical inference is not new. It has been initiated by the papers of Chernoff (1952) and Bahadur (1960), and then developped in various directions. Let us cite, among many others, Borovkov and Mogulski (1992), Groeneboom (1980), Ibragimov and Radavicius (1981), Kallenberg (1982, 1983a, 1983b), Korostelev and Leonov (1995), Nikitin (1995), Mokkadem and Pelletier (2005), and Puhalskii and Spokoiny (1998).

The idea of using MDP for the construction of confidence bands for f comes naturally when making a pointwise analysis of the confidence bands provided by Bickel and Rosenblatt (1973). Consider the univariate framework (that is, the case $d = 1$), and let C be a bounded interval of $\mathbb{R}$ on which f is assumed to be bounded away from zero. Let f_n denote the kernel estimator of f; Bickel and Rosenblatt (1973) establish the asymptotic law of $\sup_{x \in C} |f_n(x) - f(x)| / \sqrt{f(x)}$ suitably normalized. This result allows them to provide sequences of random intervals $I_n(x)$ for all $x \in C$, which satisfy the property:

$$\lim_{n \to \infty} \mathbb{P} \left(\exists x \in C, \ f(x) \notin I_n(x) \right) = \alpha.$$

For simplicity, we also denote by $I_n(x)$ the segment $\{x\} \times I_n(x)$, and say that

$$B_{n,\alpha} = \cup_{x \in C} I_n(x)$$

is a confidence band for f on C with asymptotic level $1 - \alpha \in]0, 1[$. In other words, the set of functions

$$D_{n,\alpha} = \{g : \mathbb{R} \to \mathbb{R}, \ g(x) \in I_n(x) \ \forall x \in C\}$$

is a confidence region of f with asymptotic level $1 - \alpha$. (Although $D_{n,\alpha}$ is a confidence region of the functional parameter f since $\lim_{n \to \infty} \mathbb{P} \left(f \in D_{n,\alpha} \right) = 1 - \alpha$, it gives nontrivial confidence intervals of the values $f(x)$ only for $x \in C$).

Now, a straightforward application of the central limit theorem (CLT) allows to prove that the asymptotic level of each confidence interval $I_n(x)$ ($x \in C$) whose union gives $B_{n,\alpha}$, is one. A

natural question is then to wonder at what rate the levels of the intervals $I_n(x)$ go to one. A result giving the convergence rate to zero of the sequence $\mathbb{P}\left(f(x) \notin I_n(x)\right)$ is typically a MDP result; this convergence rate is thus expected to be exponential. That is the reason why we introduce here the notion of *logarithmic asymptotic level* for confidence regions of (eventually infinite dimensional) unknown parameters.

Definition 1 *Let $\{D_n\}$ be a sequence of confidence regions of an unknown parameter θ. The logarithmic asymptotic level of $\{D_n\}$ is γ ($\gamma > 0$) with speed w_n ($w_n \to \infty$) if*

$$\lim_{n \to \infty} \frac{1}{w_n} \log \mathbb{P}(\theta \notin D_n) = -\gamma.$$

Of course, if $\{D_n\}$ has a logarithmic asymptotic level $\gamma > 0$, then the asymptotic level of $\{D_n\}$ is necessarily one.

It turns out that the sequences of confidence regions, which have a positive logarithmic asymptotic level, are often asymptotic almost sure sequences of confidence regions in the sense of the following definition.

Definition 2 *Let $\{D_n\}$ be a sequence of confidence regions of an unknown parameter θ, and let Ω denote the underlying probability space. $\{D_n\}$ is an asymptotic almost sure (or consistent) sequence of confidence regions of θ if there exists $\Omega_0 \subset \Omega$ such that:*

$$\begin{cases} \bullet & \mathbb{P}(\Omega_0) = 1 \\ \bullet & \forall \omega \in \Omega_0, \ \exists N(\omega) \text{ such that } n \geq N(\omega) \Rightarrow \theta \in D_n(\omega). \end{cases}$$

Indeed, the following proposition is a straightforward consequence of Borel and Cantelli Lemma.

Proposition 1 *Let $\{D_n\}$ be a sequence of confidence regions of an unknown parameter θ, whose logarithmic asymptotic level is $\gamma > 0$ with speed $w_n \to \infty$. If there exists $\delta \in]0, \gamma[$ such that $\sum \exp(-\delta w_n) < \infty$, then $\{D_n\}$ is an asymptotic almost sure sequence of confidence regions of θ.*

Let us come back to the pointwise analysis of the confidence bands provided by Bickel and Rosenblatt (1973). Later on, we shall prove in particular that, for all $x \in C$, the intervals $I_n(x)$ have a logarithmic asymptotic level 1 with speed $\log(1/h_n)$, where h_n is the bandwidth used for the computation of the kernel estimator f_n. So, the confidence bands $B_{n,\alpha}$ with asymptotic level $1 - \alpha < 1$ provided by Bickel and Rosenblatt are unions of confidence intervals $I_n(x)$ whose asymptotic levels equal one and whose logarithmic asymptotic levels are independent on the value of $x \in C$.

The difference between the asymptotic level of the confidence band $B_{n,\alpha}$ on the one hand and the asymptotic levels of all the confidence intervals $I_n(x)$ on the other hand is explained by the difference between the asymptotic weak behaviour of the uniform error of the kernel density estimator (given by Bickel and Rosenblatt's result) on the one hand and the asymptotic weak behaviour of the pointwise error of the kernel density estimator (given by the central limit theorem) on the other hand. Now, MDP for the nonnormalized error of the kernel density estimator have been established by Gao (2003) (see also Mokkadem, Pelletier and Worms (2005)); it turns out that, as soon as both pointwise and uniform MDP exist, the pointwise and the uniform MDP share exactly the same asymptotics. Taking this remark as a starting point, we propose, in this paper, a new approach to construct confidence bands for f based on the use of MDP for the normalized error of the kernel density estimator. This approach has several advantages:

- It allows to construct confidence bands B_n^*, which have the same width (or even a smaller width) as the confidence bands $B_{n,\alpha}$ provided by Bickel and Rosenblatt (1973), but which: (i) have an asymptotic level equal to one instead of $1 - \alpha \in]0, 1[$; (ii) share the same logarithmic asymptotic level as all the confidence intervals whose union gives B_n^*; (iii) are asymptotic almost sure confidence bands.

- In order to deal with the multivariate framework, Rosenblatt (1976) has to require the use of higher order kernels and, consequently, to impose rather stringent conditions on f; in contrast, in the MDP approach, all the dimensions are dealt with in the same way, and thus without any additional assumption neither on the density, nor on the kernel, in the case $d \geq 2$.

- Whatever the dimension d is, Bickel and Rosenblatt require the condition "f is bounded away from zero on C". On the contrary, our approach enables us to sort out the problem of providing confidence bands for f on compact sets on which f vanishes. As a matter of fact, we introduce a truncating operation, which modifies the width of our confidence bands at some points $x \in C$, but which does not affect the logarithmic asymptotic level of our confidence bands. This truncating operation also enables us to provide confidence bands for f on all $\mathbb{R}^d$. Let us mention that, in the case $d = 1$, Giné, Koltchinskii and Sakhanenko (2003, 2004) propose a slight modification of Bickel and Rosenblatt's normalization of the uniform error; this allows them to construct confidence bands on the whole line in the case f does not vanish on $\mathbb{R}$.

Our paper is now organized as follows. In Section 2, we explicit the construction of our confidence bands. Section 3 is devoted to the precise statement of our assumptions and main results. In Section 4, we discuss particular examples of applications of our main results: we first come back on the pointwise analysis of the confidence bands provided by Bickel and Rosenblatt (1973), and show how the MDP approach allows to construct more suitable confidence bands; then, we consider the problem of constructing confidence bands with smaller width. Section 5 is reserved to the proofs.

2 Construction of confidence bands based on the use of MDP

Let C be a subset of $\mathbb{R}^d$ and (v_n) be a positive nonrandom sequence that goes to infinity. In this section, we construct confidence bands for f on C with width of order v_n^{-1}. We first consider the case C is a compact set on which f is bounded away from zero, and then introduce a truncating operation, which allows to consider the general framework (f may vanish on C, C may equal $\mathbb{R}^d$).

2.1 Confidence bands on compact sets on which f is bounded away from zero

Let C be a compact set of $\mathbb{R}^d$ on which f is bounded away from zero. To construct a confidence band for f on C with width of order v_n^{-1}, we first construct, for all $x \in C$, confidence intervals of $f(x)$ with width of the same order. For that purpose, we proceed as follows.

- We estimate $f(x)$ by using the kernel estimator

$$f_n^*(x) = \frac{1}{n h_n^{*d}} \sum_{j=1}^n K\left(\frac{x - X_j}{h_n^*}\right), \qquad (1)$$

where the bandwidth (h_n^*) is a sequence of positive real numbers such that $\lim_{n \to \infty} h_n^* = 0$, $\lim_{n \to \infty} n h_n^{*d} = \infty$, and where the kernel K is a bounded nonnegative function satisfying $\int_{\mathbb{R}^d} K(z)\, dz = 1$ and $\lim_{\|z\| \to \infty} K(z) = 0$.

- The variance of $f_n^*(x)$ is equivalent (as n goes to infinity) to $(nh_n^{*d})^{-1}f(x)\kappa$ where $\kappa = \int_{\mathbb{R}^d} K^2(x)dx$; we estimate it by $(nh_n^{*d})^{-1}f_n(x)\kappa$, where f_n is the kernel estimator of f defined by

$$f_n(x) = \frac{1}{nh_n^d} \sum_{j=1}^n K\left(\frac{x - X_j}{h_n}\right), \tag{2}$$

  the bandwidth (h_n) being eventually different from (h_n^*).

- The confidence intervals for $f(x)$ $(x \in C)$ are then defined as

$$\hat{I}_n(x) = \left[f_n^*(x) - \delta\frac{\sqrt{f_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta\frac{\sqrt{f_n(x)\kappa}}{v_n}\right], \tag{3}$$

  where $\delta > 0$.

Our confidence band for f on C is finally defined by setting:

$$\hat{B}_n = \cup_{x \in C}\hat{I}_n(x). \tag{4}$$

In Section 3.2, we give conditions on the sequence (v_n) and the bandwidths (h_n) and (h_n^*), which ensure that the logarithmic asymptotic level of each interval $\hat{I}_n(x)$, $x \in C$, on the one hand, and of the confidence band $\hat{B}_n$ on the other hand, is $\delta^2/2$ with speed nh_n^{*d}/v_n^2 (see Theorems 1 and 2).

2.2 Truncating operation

In order to allow the construction of confidence bands for f on subsets C of $\mathbb{R}^d$ (eventually equal to $\mathbb{R}^d$) on which f may take the value zero, we now introduce a truncating method, which relies on the following fact. For the values of $x \in C$ for which $f_n(x)$ is "large enough", the width of the intervals $\hat{I}_n(x)$ defined in (3) is suitable; but, for the values of $x \in C$ for which $f_n(x)$ is zero, or, more generally, "close to zero", the width of the intervals $\hat{I}_n(x)$ is clearly not appropriate any more. In order to compensate for this problem which appears for "small" values of $f_n(x)$, we impose a minimum width to all the confidence intervals whose union gives the confidence band for f on C; of course, this minimum width does not affect the width of the confidence band at the points $x \in C$ for which $f_n(x)$ is "large enough".

More precisely, we introduce a sequence (ϵ_n) of positive real numbers satisfying $\lim_{n \to \infty} \epsilon_n = 0$, and define the truncating function $\tilde{T}_n$ by setting

$$\tilde{T}_n(x) = \begin{cases} f_n(x) & \text{if } f_n(x) \geq \epsilon_n, \\ \epsilon_n & \text{otherwise.} \end{cases} \tag{5}$$

For each $x \in C$, we set

$$\check{I}_n(x) = \left[f_n^*(x) - \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\right] \tag{6}$$

and finally define $\check{B}_n$ as

$$\check{B}_n = \cup_{x \in C}\check{I}_n(x). \tag{7}$$

In Section 3.3, we give conditions on (ϵ_n), which ensure that the logarithmic asymptotic level of the confidence band $\check{B}_n$ is $\delta^2/2$ with speed nh_n^{*d}/v_n^2 (see Theorem 3 in the case C is a compact set, and Theorem 4 in the case $C = \mathbb{R}^d$). In other words, the logarithmic asymptotic level of the

confidence band $\hat{B}_n$ defined in (4) is not affected by the introduction of this truncating method.

From a practical point of view, it seems more realistic to take the width of the largest confidence interval into account in the truncating operation, that is, to introduce the quantity $\sup_{x \in C} f_n(x)$ in the definition of the truncating function. For that purpose, let the sequence (ϵ_n) satisfy the additional condition $\epsilon_n \leq 1$ for all n, and define the function T_n by setting

$$T_n(x) = \begin{cases} f_n(x) & \text{if } f_n(x) \geq \epsilon_n[\sup_{x \in C} f_n(x)] \\ \epsilon_n[\sup_{x \in C} f_n(x)] & \text{otherwise.} \end{cases} \tag{8}$$

In Section 3.3, we establish that when the parameter $\tilde{T}_n(x)$ is replaced by $T_n(x)$ in the intervals $\check{I}_n(x)$, the logarithmic asymptotic level of the confidence band $\check{B}_n$ remains unchanged (see Corollary 1 in the case C is a compact set, and Corollary 2 in the case $C = \mathbb{R}^d$).

3   Assumptions and Main Results

3.1   Assumptions

Before stating our assumptions, let us first define the *covering number condition*. Let Q be a probability on $\mathbb{R}^d$ and $\mathcal{F} \subset \mathcal{L}_2(Q)$ be a class of Q-integrable functions. The covering number (see Pollard (1984)) is the smallest value $N_2(\epsilon, Q, \mathcal{F})$ of m for which there exist m functions $g_1, \ldots, g_m \in \mathcal{L}_2(Q)$ such that

$$\min_{i \in \{1, \ldots, m\}} \|f - g_i\|_{\mathcal{L}_2(Q)} \leq \epsilon \ \ \forall f \in \mathcal{F}$$

(if no such m exists, $N_2(\epsilon, Q, \mathcal{F}) = \infty$). Now, let Λ be a bounded and integrable function on $\mathbb{R}^d$, and let $\mathcal{F}(\Lambda)$ be the class of functions defined by

$$\mathcal{F}(\Lambda) = \left\{ z \mapsto \Lambda \left(\frac{x - z}{h} \right), \ \ h > 0, \ \ x \in \mathbb{R}^d \right\}. \tag{9}$$

Λ *is said to satisfy the covering number condition if there exist $A > 0$ and $v > 0$ such that, for any probability Q on $\mathbb{R}^d$ and any $\epsilon \in]0, 1[$,*

$$N_2(\epsilon \|\Lambda\|_\infty, Q, \mathcal{F}(\Lambda)) \leq \left(\frac{A}{\epsilon} \right)^v. \tag{10}$$

The classes which satisfy (10) are often called Vapnik-Chervonenkis classes. When $d = 1$, the real valued kernels with bounded variations satisfy the covering number condition (see Pollard (1984)). Some examples of multivariate kernels satisfying the covering number condition are the following :

- the kernels defined as $K(x) = \psi(\|x\|)$, where ψ is a real valued function with bounded variations (see Nolan and Pollard (1987)).

- the kernels defined as $K(x) = \prod_{i=1}^d K_i(x_i)$ where the K_i, $1 \leq i \leq d$, are real valued functions with bounded variations (this follows from Lemma A1 in Einmahl and Mason (2000)).

We can now state our assumptions.

$X_1, \ldots, X_n$ are *i.i.d.* $\mathbb{R}^d$-valued random vectors with bounded probability density f. The kernel estimators f_n and f_n^* of f are defined in (2) and (1) respectively, and the bandwidths (h_n) and (h_n^*) are two sequences of positive real numbers such that

$$h_n \to 0 \ \ \text{and} \ \ h_n^* \to 0.$$

The assumptions to which we will refer in the sequel are the following.

(A1) K is a bounded and nonnegative function on $\mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} K(z)dz = 1, \quad \int_{\mathbb{R}^d} z_j K(z)dz = 0 \ \forall j \in \{1,\ldots,d\}, \quad \text{and} \quad \int_{\mathbb{R}^d} \|z\|^2 |K(z)|dz < \infty.$$

(A2) K is Hölder-continuous on $\mathbb{R}^d$ and satisfies the covering number condition.

(A3) f is twice differentiable on $\mathbb{R}^d$, $\sup_{x \in \mathbb{R}^d} \|\nabla f(x)\| < \infty$, and $\sup_{x \in \mathbb{R}^d} \|D^2 f(x)\| < \infty$.

(A4) There exists $q > 0$ such that $z \mapsto \|z\|^q f(z)$ is a bounded function on $\mathbb{R}^d$.

Let us recall the notation

$$\kappa = \int_{\mathbb{R}^d} K^2(z)dz.$$

3.2 Confidence regions without truncating

Let (v_n) be a sequence satisfying $v_n \to \infty$. The object of our first two theorems is to specify the logarithmic asymptotic level of the sequences of confidence intervals and of confidence bands defined in (3) and (4) respectively.

Theorem 1 *Assume* (A1) *holds, set* $x \in \mathbb{R}^d$ *such that* $f(x) \neq 0$, *and assume that* f *is twice differentiable at* x. *Moreover, assume that* (h_n), (h_n^*) *and* (v_n) *satisfy the conditions*

$$\frac{nh_n^{*d}}{v_n^2} \to \infty, \quad v_n h_n^{*2} \to 0, \quad \text{and} \quad \frac{v_n^2 h_n^d}{h_n^{*d}} \to \infty. \tag{11}$$

Then, for any $\delta > 0$, *we have*

$$\lim_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right) = -\frac{\delta^2}{2}.$$

Moreover, if the additional condition

$$\frac{nh_n^{*d}}{v_n^2 \log(1/h_n^*)} \to \infty$$

holds, then the sequence of intervals

$$\left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right]$$

is an asymptotic almost sure sequence of confidence intervals of $f(x)$.

Theorem 2 *Let* (A1) − (A3) *hold and assume that* f *is bounded away from zero on a compact set* C. *Moreover, assume that* (h_n), (h_n^*) *and* (v_n) *satisfy the conditions*

$$\frac{nh_n^{*d}}{v_n^2 \log(1/h_n^*)} \to \infty, \quad v_n h_n^{*2} \to 0, \quad \text{and} \quad \frac{v_n^2 h_n^d}{h_n^{*d}} \to \infty. \tag{12}$$

Then, for any $\delta > 0$, *we have*

$$\lim_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(\exists x \in C, \ \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right) = -\frac{\delta^2}{2}.$$

Moreover, the sequence of sets of functions

$$D_n = \left\{ g : \mathbb{R}^d \to \mathbb{R}, \; |g(x) - f_n^*(x)| \leq \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \; \forall x \in C \right\}$$

is an asymptotic almost sure sequence of confidence regions of f.

Comments on Theorems 1 and 2

1) The regularity assumption on f in Theorem 1 is usually required to establish a CLT for $f_n^*(x)$ in the case when f_n^* is defined with a two-order kernel; the assumptions on f in Theorem 2 are weaker than those required by Bickel and Rosenblatt (1973) to establish the asymptotic distribution of the normalized uniform error of the kernel density estimator.

2) The condition $v_n h_n^{*2} \to 0$ (together with the regularity assumption on f) ensures that the bias of f_n^* does not interfer. In the case f is only once differentiable on $\mathbb{R}^d$, this condition must be replaced by $v_n h_n^* \to 0$ for Theorems 1 and 2 hold.

3) The main tool used to prove Theorem 1 is pointwise MDP established for the normalized error of the kernel density estimator, whereas the demonstration of Theorem 2 relies on the use of uniform MDP; that is the reason why the conditions (12) in Theorem 2 are slightly stronger than the conditions (11) of Theorem 1. Now, as soon as the conditions of Theorem 2 hold, pointwise and uniform MDP give exactly the same asymptotics. This unity in pointwise and uniform MDP differs from the gap there exists between the nature of the asymptotic distribution of the normalized pointwise error of the kernel density estimator (given by the CLT) on the one hand, and of the normalized uniform error of the kernel density estimator (given by Bickel and Rosenblatt (1973)) on the other hand.

3.3 Confidence regions with truncating

The next theorem allows to set up confidence bands for f on compact sets C on which f may take the value zero. Let the positive real-valued sequences (h_n), (h_n^*), (v_n) and (ϵ_n) satisfy the following conditions

$$\begin{cases} \epsilon_n \to 0, \quad \dfrac{h_n^*}{\epsilon_n} \to 0, \quad \dfrac{h_n^2}{\epsilon_n} \to 0, \\[2ex] v_n \epsilon_n^{3/2} \to \infty, \quad \dfrac{v_n h_n^{*2}}{\sqrt{\epsilon_n}} \to 0, \quad \dfrac{n h_n^{*d}}{v_n^2 \log(1/h_n^*)} \to \infty, \quad \dfrac{v_n^2 h_n^d \epsilon_n^2}{h_n^{*d}} \to \infty, \end{cases} \tag{13}$$

and let $\tilde{T}_n$ be the function defined by (5).

Theorem 3 *Let $(A1) - (A3)$ hold, assume that there exists $x \in C$ such that $f(x) \neq 0$, and that the sequences (h_n), (h_n^*), (v_n) and (ϵ_n) satisfy (13). Then, the conclusions of Theorem 2 still hold when $f_n(x)$ is replaced by $\tilde{T}_n(x)$.*

Let T_n be the function defined by (8) with $\epsilon_n \leq 1$; with the help of Theorem 3, we will prove the following result.

Corollary 1 *Let $(A1) - (A3)$ hold, assume that there exists $x \in C$ such that $f(x) \neq 0$, and that (h_n), (h_n^*), (v_n), and (ϵ_n) satisfy (13). Then, the conclusions of Theorem 2 still hold when $f_n(x)$ is replaced by $T_n(x)$.*

The extension of Theorem 3 and Corollary 1 to the case $C = \mathbb{R}^d$ holds under the additional assumption $(A4)$.

Theorem 4 *Let $(A1) - (A4)$ hold, and assume (h_n), (h_n^*), (v_n) and (ϵ_n) satisfy (13). Then, the conclusions of Theorem 2 still hold when $f_n(x)$ is replaced by $\tilde{T}_n(x)$ and C by $\mathbb{R}^d$.*

Let T_n be defined by (8) with $\epsilon_n \leq 1$ and $C = \mathbb{R}^d$.

Corollary 2 *Let $(A1) - (A4)$ hold, and assume (h_n), (h_n^*), (v_n), and (ϵ_n) satisfy (13). Then, the conclusions of Theorem 2 still hold when $f_n(x)$ is replaced by $T_n(x)$ and C by $\mathbb{R}^d$.*

Remark Let us mention that Corollaries 1 and 2 also hold when the sequence (ϵ_n) is constant $(\epsilon_n = \epsilon \in]0, 1]$ for all n); in the case $\epsilon_n = 1$, the width of the confidence bands does not depend on the point $x \in C$.

4 Particular cases

In this section, we first give a pointwise analysis of the confidence bands provided by Bickel and Rosenblatt (1973), and show how the MDP approach allows to modify these confidence bands in order to obtain confidence bands whose width is of the same order as the one of Bickel and Rosenblatt's confidence bands, but whose asymptotic level equals one instead of $1 - \alpha < 1$; in particular, we explicit the choices of the parameters (h_n^*) and (ϵ_n), which give the best convergence rate to one of the level of these modified confidence bands. Then, we consider the problem of constructing confidence bands, which are thinner than those provided by Bickel and Rosenblatt, but whose level converges to one slower than the level of the modified Bickel and Rosenblatt's confidence bands does. We give two possible choices of (h_n^*), which both correspond to the case our confidence bands are centered at an optimal kernel estimator of f; for the first choice, the optimality is according to the L^2 criterion, and, for the second one, to the L^∞ criterion.

4.1 On Bickel and Rosenblatt's confidence bands

4.1.1 Pointwise analysis of Bickel and Rosenblatt's confidence bands

Set $d = 1$ and let $C = [c_1, c_2]$ be a bounded interval of $\mathbb{R}$. Bickel and Rosenblatt (1973) construct confidence bands for f on C with asymptotic level $1 - \alpha \in]0, 1[$ in the case when: (i) f is bounded away from zero on C; (ii) the kernel K is chosen absolutely continuous on $\mathbb{R}$ and such that $\int_{\mathbb{R}} K'^2(t)dt \neq 0$; (iii) the bandwidth h_n used for the computation of f_n is chosen equal to (n^{-a}) with $a \in]\frac{1}{5}, \frac{1}{2}[$. Their confidence bands are constructed as follows.

Set $\alpha \in]0, 1[$, z_α such that $\exp(-2\exp(-z_\alpha)) = 1 - \alpha$, and, for all $x \in C$,

$$
\begin{aligned}
I_n(x) = & \left[f_n(x) - \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}}\sqrt{\log(1/h_n)}\left(\sqrt{2} + u_n + \frac{z_\alpha}{\sqrt{2}\log(1/h_n)} \right) \right. ; \\
& \left. f_n(x) + \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}}\sqrt{\log(1/h_n)}\left(\sqrt{2} + u_n + \frac{z_\alpha}{\sqrt{2}\log(1/h_n)} \right) \right]
\end{aligned}
\tag{14}
$$

with

$$
u_n = \frac{1}{\sqrt{2}\log(1/h_n)}\left\{ \log\left[\frac{1}{2\pi}\sqrt{\frac{\int_{\mathbb{R}} K'^2(t)dt}{\kappa}} \right] + \log\left[\frac{c_2 - c_1}{\pi} \right] \right\}.
\tag{15}
$$

8

Bickel and Rosenblatt (1973) prove that

$$B_{n,\alpha} = \cup_{x \in C} I_n(x)$$

is then a confidence band for f on C with asymptotic level $1 - \alpha$. A straightforward application of the CLT ensures that, for each $x \in C$, the asymptotic level of $I_n(x)$ equals one. Now, Theorem 1 allows to specify the convergence rate of the asymptotic level of the confidence intervals $I_n(x)$ toward one. More precisely, the application of Theorem 1 with $(h_n) \equiv (h_n^*)$ and $(v_n) \equiv (\sqrt{nh_n/\log(1/h_n)})$, together with a continuity argument, ensure that **the logarithmic asymptotic level of each confidence interval $I_n(x)$ is 1 with speed** $\log(1/h_n)$.

The difference between the asymptotic level of the confidence band $B_{n,\alpha}$ and the asymptotic levels of all the confidence intervals $I_n(x)$ is explained by the difference between the asymptotic weak behaviour of the uniform error of f_n (given by Bickel and Rosenblatt's result) and the asymptotic weak behaviour of the pointwise error of f_n (given by the central limit theorem). Adopting the MDP point of view, it is note-worthy that this phenomenon corresponds to the case pointwise MDP hold, but not uniform MDP. As a matter of fact, when $d = 1$, the sequence $(v_n) \equiv (\sqrt{nh_n/\log(1/h_n)})$ fulfills the conditions (11) required by Theorem 1, but not the slightly stronger conditions (12) imposed by Theorem 2.

4.1.2   Improvement of Bickel and Rosenblatt's confidence bands

The aim of this section is to show how the MDP approach allows to improve the confidence bands provided by Bickel and Rosenblatt (1973). In a first part, we introduce a translation, which allows to provide confidence bands that have the same width as the confidence bands provided by Bickel and Rosenblatt, but which have a better asymptotic level. In a second part, we give a simplification of these translated confidence bands, which does affect neither their width order, nor their logarithmic asymptotic level. Then, we show how we can get rid of the condition f is bounded away from zero on C. Finally, we give the extension to the multivariate framework.

Confidence bands translation   We consider here Bickel and Rosenblatt's framework, that is, the case $d = 1$, $C = [c_1, c_2]$, and f is bounded away from zero on C.

Set $(h_n) \equiv (n^{-a})$ with $a \in]\frac{1}{5}, \frac{1}{2}[$, let f_n, u_n and z_α be defined in the same way as in Section 4.1.1, and (h_n^*) be a bandwidth satisfying the conditions

$$n^a h_n^* \to \infty \quad \text{and} \quad \frac{n^{1-a} h_n^{*4}}{\log n} \to 0. \tag{16}$$

Moreover, let f_n^* be the kernel estimator of f defined with the bandwidth h_n^*, and, for each $x \in C$, set

$$
\begin{aligned}
I_n^*(x) \;=\; & \left[f_n^*(x) - \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}} \sqrt{\log(1/h_n)} \left(\sqrt{2} + u_n + \frac{z_\alpha}{\sqrt{2}\log(1/h_n)} \right) \; ; \right. \\
& \left. f_n^*(x) + \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}} \sqrt{\log(1/h_n)} \left(\sqrt{2} + u_n + \frac{z_\alpha}{\sqrt{2}\log(1/h_n)} \right) \right].
\end{aligned}
$$

Note that, for each x in C, $I_n^*(x)$ is the translation of the confidence interval $I_n(x)$ (defined in (14)) from the quantity $f_n^*(x) - f_n(x)$.

The application of Theorem 1 (with $d = 1$ and $(v_n) \equiv (\sqrt{nh_n/\log(1/h_n)})$), together with a continuity argument, ensure that, **for each x in C, the logarithmic asymptotic level of $I_n^*(x)$ is equal to 1 with speed $h_n^* \log(1/h_n)/h_n$.** Let us underline that the speed obtained for $I_n^*(x)$ is faster than the speed obtained for $I_n(x)$; in other words, the levels of the translated intervals $I_n^*(x)$ go to one faster than the levels of the intervals $I_n(x)$. This is explained by the fact that the translated intervals $I_n^*(x)$ are centered at the point $f_n^*(x)$ rather than at the point $f_n(x)$, and, in view of the conditions (16), the estimator $f_n^*(x)$ converges to $f(x)$ faster than the estimator $f_n(x)$ does.

Now, set
$$B_n^* = \cup_{x \in C} I_n^*(x).$$
The application of Theorem 2 (together with a continuity argument) ensures that B_n^* **is a confidence band for f on C whose logarithmic asymptotic level equals 1 with speed $h_n^* \log(1/h_n)/h_n$.**

The confidence band B_n^*, which is just the translation of $B_{n,\alpha}$ from the quantity $f_n^* - f_n$, has thus the following advantages:

- It has the same width, at each point $x \in C$, as the confidence band $B_{n,\alpha}$ provided by Bickel and Rosenblatt.

- Its asymptotic level is one instead of being $1 - \alpha < 1$.

- The logarithmic asymptotic level of B_n^* is the same as the logarithmic asymptotic levels of all the intervals $I_n^*(x)$ whose union gives B_n^*, and the intervals $I_n^*(x)$ themselves have a better logarithmic asymptotic level than the intervals $I_n(x)$ whose union gives $B_{n,\alpha}$.

Let us mention that the advisable choice of the bandwidth (h_n^*) is $(h_n^*) \equiv (n^{-(1-a)/4})$, where a is the parameter which defines (h_n). As a matter of fact, among the sequences $(h_n^*) \equiv (n^{-b})$ which satisfy (16), it is the choice that maximizes the speed $h_n^* \log(1/h_n)/h_n$ (which then equals $n^{(5a-1)/4} \log n$). Let us underline that the condition $a > 1/5$ implies $(1 - a)/4 < 1/5$. For this optimal choice of the bandwidth (h_n^*), the confidence band B_n^* (respectively the confidence interval $I_n^*(x)$) is thus centered at an estimator f_n^* (respectively $f_n^*(x)$) whose convergence rate is given by the convergence rate of its bias term. Consequently, B_n^* (respectively $I_n^*(x)$) cannot be compared with a confidence band (respectively confidence interval) centered at f_n^* (respectively $f_n^*(x)$) and provided by Bickel and Rosenblatt's result (respectively by the central limit theorem). The surprising aspect of this result is that this optimal choice of (h_n^*) depends on the choice of the bandwidth (h_n) and is never the choice for which the estimator f_n^* converges at the optimal rate.

Simplification of the translated confidence bands The parameters u_n (which depends on the length of the interval C) and z_α (which depends on the asymptotic level α), which appear in the definitions of the intervals $I_n(x)$ and $I_n^*(x)$, play a crucial role in Bickel and Rosenblatt's approach. However, they do not have any effect in the MDP approach. That is the reason why we propose here a simplification of the definition of the confidence band B_n^*. More precisely, we set
$$B_n^{**} = \cup_{x \in C} I_n^{**}(x),$$
where, for each $x \in C$,
$$I_n^{**}(x) = \left[f_n^*(x) - \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}} \sqrt{\log(1/h_n)} \sqrt{2} \; ; \; f_n^*(x) + \frac{\sqrt{f_n(x)\kappa}}{\sqrt{nh_n}} \sqrt{\log(1/h_n)} \sqrt{2} \right].$$

A straightforward application of Theorems 1 and 2 ensures that **the logarithmic asymptotic levels of $I_n^{**}(x)$ for all $x \in C$ and of B_n^{**} equal 1 with speed** $h_n^* \log(1/h_n)/h_n$**.** In particular, we see that this simplification does not affect the logarithmic asymptotic level.

Confidence bands truncating Although the simplified confidence band B_n^{**} seems very convenient to use, it suffers from the same drawback as the confidence band $B_{n,\alpha}$ proposed by Bickel and Rosenblatt (1973): its use is conditionned to the fact that the density f is bounded away from zero on C. In order to allow the construction of confidence bands for f on intervals C on which f may take the value zero, we now introduce the truncated confidence band defined as:

$$B_n^{***} = \cup_{x \in C} I_n^{***}(x),$$

where, for each $x \in C$,

$$I_n^{***}(x) = \left[f_n^*(x) - \frac{\sqrt{T_n(x)\kappa}}{\sqrt{nh_n}}\sqrt{\log(1/h_n)}\sqrt{2} \; ; \; f_n^*(x) + \frac{\sqrt{T_n(x)\kappa}}{\sqrt{nh_n}}\sqrt{\log(1/h_n)}\sqrt{2} \right],$$

T_n being the truncating function defined in (8). A straightforward application of Corollary 1 (respectively of Corollary 2) in the case C is a compact set (respectively in the case $C = \mathbb{R}$) ensures that, if $(\epsilon_n) \equiv (\log n)^{-e}$ with $e \in]0,1[$, then **the logarithmic asymptotic level of B_n^{***} is 1 with speed** $h_n^* \log(1/h_n)/h_n$**.** In other words, the truncating operation, which allows the construction of confidence bands for the density on compact sets on which f vanishes or on the whole line, does not affect the logarithmic asymptotic level.

Let us underline that the advantage of truncating is not only to enable the construction of confidence bands for f on intervals on which f may take the value zero. Even in the case f is bounded away from zero on C, truncating gives, in practice, much better results as soon as the length of the interval C is large.

The multivariate framework As mentionned in the introduction, the problem of constructing confidence bands when the probability density f is defined on $\mathbb{R}^d$ has been considered by Rosenblatt (1976). His approach consists in an extension to the d-dimensional case ($d > 1$) of the results obtained by Bickel and Rosenblatt (1973). However, in order to enable the construction of confidence bands for f on a compact set C on $\mathbb{R}^d$ (on which f is bounded away from zero), Rosenblatt (1976) requires the use of kernels of order $k > d(d+2)/2$, and, consequently, imposes rather stringent conditions on f. On the opposite, with the MDP approach, all the dimensions are dealt with in the same way. More precisely, let the bandwidths (h_n) and (h_n^*) be defined as $(h_n) \equiv (n^{-a})$ with $a \in]\frac{1}{d+4}, \frac{d+4}{d(d+8)}[$ and $(h_n^*) \equiv (n^{-(1-ad)/4})$, the sequence (ϵ_n) as $(\epsilon_n) \equiv (\log n)^{-e}$ with $e \in]0,1[$, and set, for each $x \in C$,

$$I_n^{***}(x) = \left[f_n^*(x) - \frac{\sqrt{T_n(x)\kappa}}{\sqrt{nh_n^d}}\sqrt{\log(1/h_n)}\sqrt{2} \; ; \; f_n^*(x) + \frac{\sqrt{T_n(x)\kappa}}{\sqrt{nh_n^d}}\sqrt{\log(1/h_n)}\sqrt{2} \right],$$

where the truncating function T_n is defined in (8). A straightforward application of Corollary 1 (respectively of Corollary 2) in the case C is a compact set (respectively in the case $C = \mathbb{R}^d$) ensures that, **the logarithmic asymptotic level of the confidence band $B_n^{***} = \cup_{x \in C} I_n^{***}(x)$ is 1 with speed** $h_n^{*d} \log(1/h_n)/h_n^d$. Let us mention that this implies the existence of two positive functions λ_1^+ and λ_1^- which go to infinity with a logarithmic rate, and such that

$$\exp\left(-\frac{\delta^2}{2} n^{([d+4]a-1)d/4} \lambda_1^-(n) \right) \leq \mathbb{P}\left(\exists x \in C, \; f(x) \notin I_n^{***}(x) \right) \leq \exp\left(-\frac{\delta^2}{2} n^{([d+4]a-1)d/4} \lambda_1^+(n) \right).$$

4.2 Thinner confidence bands

The width order of the confidence band B_n^{***} is $(\log n)^{1/2} n^{-b}$ with $b < 2/(d+4)$; this width might seem too large, and thinner confidence bands might be prefered, although the convergence rate to 1 of their asymptotic level is slower.

The smallest possible width of confidence bands whose width does not depend on $\sqrt{f(x)}$ and whose asymptotic level equals $1 - \alpha < 1$ is, according to the minimax theory, $M[(\log n)/n]^{2/(d+4)}$ where the constant M depends on some known bound of $\|f\|_\infty$ and $\|f''\|_\infty$ (see Ibragimov and Hasminskii (1981), Donoho and Liu (1991), Donoho (1994), and Tsybakov (2004)). This optimal width can not be reached in the case the width of the confidence bands depends on $\sqrt{f(x)}$; as a matter of fact, for a large class of densities (which includes the standard Gaussian density), the sequence $[n/\log n]^{2/(d+4)} \|(f_n - f)/\sqrt{f}\|_\infty$ is known to be not stochastically bounded (see Giné and Guillou (2002), pp. 918).

In this section, we give two examples of choices of the parameters (h_n^*), (v_n), (h_n), and (ϵ_n), which lead to confidence bands whose width order is close to $[(\log n)/n]^{2/(d+4)}$.

- Set $(h_n^*) \equiv (c^* n^{-1/(d+4)})$ with $c^* > 0$; this choice corresponds to the case the confidence bands provided by the MDP approach are centered at the kernel estimator, which minimizes the (integrated) mean squared error. For this choice of bandwidth, the sequence (v_n) can be chosen as:

$$(v_n) \equiv \left(v^* \frac{n^{2/(d+4)}}{(\log n)^a} \right) \quad \text{with} \quad v^* > 0 \quad \text{and} \quad a > \frac{1}{2},$$

  the sequence (h_n) can be chosen equal to (h_n^*) or to $(c[n/\log n]^{-1/(d+4)})$ with $c > 0$, and the sequence (ϵ_n) as:

$$(\epsilon_n) \equiv \left(\epsilon^* (\log n)^{-e} \right) \quad \text{with} \quad \epsilon^* > 0 \quad \text{and} \quad e < 2a.$$

  The application of Corollary 1 (in the case C is a compact set) or of Corollary 2 (in the case $C = \mathbb{R}^d$) ensures that the logarithmic asymptotic level of the confidence bands defined as $B_n = \cup_{x \in C} I_n(x)$ with

$$I_n(x) = \left[f_n^*(x) - \delta \frac{\sqrt{T_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta \frac{\sqrt{T_n(x)\kappa}}{v_n} \right] \tag{17}$$

  is then equal to $\delta^2/2$ with speed nh_n^{*d}/v_n^2. Consequently, there exist two positive functions λ_2^+ and λ_2^- which go to infinity with a logarithmic rate, and such that

$$n^{-\frac{\delta^2}{2} \lambda_2^-(n)} \leq \mathbb{P}\left(\exists x \in C, \; f(x) \notin I_n(x) \right) \leq n^{-\frac{\delta^2}{2} \lambda_2^+(n)}.$$

- Set $(h_n^*) \equiv (c^* [n/\log n]^{-1/(d+4)})$ with $c^* > 0$; this choice corresponds to the case the confidence bands are centered at the kernel estimator, which minimizes the uniform error. For this choice of bandwidth, we can construct confidence bands whose width is arbitrarily close to $[(\log n)/n]^{2/(d+4)}$, by choosing the sequence (v_n) as

$$(v_n) \equiv \left(v^* \left[\frac{n}{\log n} \right]^{2/(d+4)} \frac{1}{(\log \log n)^a} \right) \quad \text{with} \quad v^* > 0 \quad \text{and} \quad a > 0,$$

  the sequence (h_n) equal to (h_n^*), and the sequence (ϵ_n) as

$$(\epsilon_n) \equiv \left(\epsilon^* (\log \log n)^{-e} \right) \quad \text{with} \quad \epsilon^* > 0 \quad \text{and} \quad e < 2a.$$

The application of Corollaries 1 and 2 ensures that logarithmic asymptotic level of the confidence bands defined as $B_n = \cup_{x \in C} I_n(x)$ with $I_n(x)$ defined in (17) is then equal to $\delta^2/2$ with speed nh_n^{*d}/v_n^2. Accordingly,

$$(\log n)^{-\frac{\delta^2}{2}\lambda_3^-(n)} \leq \mathbb{P}\left(\exists x \in C, \; f(x) \notin I_n(x)\right) \leq (\log n)^{-\frac{\delta^2}{2}\lambda_3^+(n)}$$

where λ_3^+ and λ_3^- are two positive functions, which go to infinity with a rate in $\log \log$.

Let us finally mention that, in all the previous examples, the truncating function T_n can be replaced by the function $\tilde{T}_n$ defined in (5).

5 Proofs

We first give a unified proof for all the almost sure parts of our results in Section 5.1. Then, Theorem 1 is proved in Section 5.2, Theorems 2, 3, and 4 in Section 5.3, and Corollaries 1 and 2 in Section 5.4.

5.1 Unified proof for all the almost sure parts of our results

The proof relies on the use of both conditions

$$v_n h_n^{*2} \to 0 \quad \text{and} \quad \frac{nh_n^{*d}}{v_n^2 \log(1/h_n^*)} \to \infty.$$

Set $\gamma = \delta^2/2$, $w_n = nh_n^{*d}/v_n^2$, $\rho \in]0, 1/(d+4)[$, and $M > 1/\rho$. On the one hand, the condition $v_n h_n^{*2} \to 0$ implies that, for n large enough,

$$\exp(-\gamma w_n/2) \leq \exp\left[\frac{-\gamma nh_n^{*(d+4)}}{2}\right].$$

On the other hand, the condition $nh_n^{*d}/[v_n^2 \log(1/h_n^*)] \to \infty$ implies that, for n large enough, $nh_n^{*d}/v_n^2 \geq 2M \log(1/h_n^*)/\gamma$, and thus

$$\exp(-\gamma w_n/2) \leq h_n^{*M}.$$

It follows that

$$\exp(-\gamma w_n/2) \leq \begin{cases} \exp\left(-\gamma n^{1-(d+4)\rho}/2\right) & \text{if } h_n^* \geq n^{-\rho} \\ n^{-M\rho} & \text{if } h_n^* \leq n^{-\rho}, \end{cases}$$

and thus $\sum_n \exp(-\gamma w_n/2) < \infty$. The almost sure parts of our results then follow from the application of Proposition 1.

5.2 Proof of Theorem 1

Set $\delta > 0$ and $\eta \in]0, 1[$. On the one hand, we have

$$\mathbb{P}\left(f(x) \notin \left[f_n^*(x) - \delta\frac{\sqrt{f_n(x)\kappa}}{v_n}; \; f_n^*(x) + \delta\frac{\sqrt{f_n(x)\kappa}}{v_n}\right]\right)$$

$$\leq \; \mathbb{P}\left[v_n|f_n^*(x) - f(x)| > \delta\sqrt{f_n(x)\kappa} \text{ and } \frac{f_n(x)}{f(x)} > \frac{1}{1+\eta}\right] + \mathbb{P}\left[\frac{f_n(x)}{f(x)} \leq \frac{1}{1+\eta}\right]$$

$$\leq \; \mathbb{P}\left[v_n|f_n^*(x) - f(x)| > \frac{\delta\sqrt{f(x)\kappa}}{\sqrt{1+\eta}}\right] + \mathbb{P}\left[f(x) - f_n(x) \geq \frac{\eta f(x)}{1+\eta}\right].$$

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Now, Theorem 4 in Mokkadem, Pelletier and Worms (2005) ensures that

$$\lim_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[v_n |f_n^*(x) - f(x)| > \frac{\delta\sqrt{f(x)\kappa}}{\sqrt{1+\eta}} \right] = \frac{-\delta^2}{2(1+\eta)},$$

and, since $v_n^2 h_n^d / h_n^{*d} \to \infty$, the application of Corollary 1 in Mokkadem, Pelletier and Worms (2005) gives

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[f(x) - f_n(x) \geq \frac{\eta f(x)}{1+\eta} \right] \leq \limsup_{n \to \infty} \frac{v_n^2 h_n^d}{h_n^{*d}} \left[\frac{1}{nh_n^d} \log \mathbb{P}\left[|f(x) - f_n(x)| \geq \frac{\eta f(x)}{1+\eta} \right] \right].$$
$$= -\infty.$$

We thus deduce that

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right) \leq \frac{-\delta^2}{2(1+\eta)}. \qquad (18)$$

On the other hand, we note that

$$\mathbb{P}\left(f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right)$$

$$\geq \mathbb{P}\left[v_n |f_n^*(x) - f(x)| > \delta\sqrt{f(x)\kappa}\sqrt{\frac{f_n(x)}{f(x)}} \; \text{ and } \; \sqrt{\frac{f_n(x)}{f(x)}} \leq \sqrt{1+\eta} \right]$$

$$\geq \mathbb{P}\left[v_n |f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)f(x)\kappa} \; \text{ and } \; \sqrt{\frac{f_n(x)}{f(x)}} \leq \sqrt{1+\eta} \right]$$

$$\geq \mathbb{P}\left[v_n |f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)f(x)\kappa} \right] - \mathbb{P}\left[f_n(x) > (1+\eta)f(x) \right]$$

$$\geq \mathbb{P}\left[v_n |f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)f(x)\kappa} \right] - \mathbb{P}\left[|f_n(x) - f(x)| > \eta f(x) \right],$$

and the application of Corollary 1 and Theorem 4 in Mokkadem, Pelletier and Worms (2005) leads to

$$\liminf_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \; ; \; f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right) \geq \frac{-\delta^2(1+\eta)}{2}. \qquad (19)$$

Since η can be taken arbitrarily close to zero, Theorem 1 is a straightforward consequence of (18) and (19).

5.3 Proof of Theorems 2, 3, and 4

The proof of Theorems 2, 3, and 4 will require the application of Lemmas 1 and 2 below. We first state these Lemmas, whose proof is postponed in the appendix (see Section 6.1). Then, we prove Theorems 4, 3, and 2 in Sections 5.3.1, 5.3.2, and 5.3.3 respectively.

Let C_n be a sequence of compact sets of $\mathbb{R}^d$ and set $w_n = \sup\{\|x\|, \ x \in C_n\}$. Moreover, set $\xi \in]0,1[, \zeta > 1$, and

$$\tilde{U}_n = \{x \in C_n, \ f_n(x) \geq \epsilon_n\}, \qquad (20)$$
$$\tilde{W}_n = \{x \in C_n, \ f_n(x) < \epsilon_n\}, \qquad (21)$$
$$U_n(\xi) = \{x \in C_n, \ f(x) \geq \xi\epsilon_n\}, \qquad (22)$$
$$W_n(\zeta) = \{x \in C_n, \ f(x) \leq \zeta\epsilon_n\}. \qquad (23)$$

Lemma 1 *Assume that (A1)-(A3) hold, and that (v_n), (h_n), (h_n^*), (ϵ_n) satisfy (13). Moreover, assume that (w_n) fulfills the condition*

$$\lim_{n \to \infty} \frac{v_n^2 \log w_n}{n h_n^{*d}} = 0. \tag{24}$$

Then, for any $\delta > 0$,

$$\limsup_{n \to \infty} \frac{v_n^2}{n h_n^{*d}} \log \mathbb{P} \left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \quad and \quad \inf_{x \in U_n(\xi)} f_n(x) > 0 \right] \leq -\frac{\delta^2}{2}.$$

Lemma 2 *Assume that (A1)-(A3) hold, and that (v_n), (h_n), (h_n^*), (ϵ_n) and (w_n) satisfy (13) and (24). For any $\delta > 0$,*

$$\limsup_{n \to \infty} \frac{v_n^2}{n h_n^{*d}} \log \mathbb{P} \left[\sup_{x \in W_n(\zeta)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right] \leq -\delta^2 \left(1 - \frac{\zeta}{2} \right).$$

5.3.1 Proof of Theorem 4

To prove Theorem 4, we first establish the upper bound

$$\limsup_{n \to \infty} \frac{v_n^2}{n h_n^{*d}} \log \mathbb{P} \left(\exists x \in \mathbb{R}^d, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right) \leq -\frac{\delta^2}{2}, \quad (25)$$

and then prove the lower bound

$$\liminf_{n \to \infty} \frac{v_n^2}{n h_n^{*d}} \log \mathbb{P} \left(\exists x \in \mathbb{R}^d, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right) \geq -\frac{\delta^2}{2}. \quad (26)$$

Throughout the proof, we set $C_n = \left\{ x \in \mathbb{R}^d, \ \|x\| \leq \epsilon_n^{-2/q} \right\}$ (we thus have $w_n = \epsilon_n^{-2/q}$).

Proof of the upper bound (25) We have

$$\mathbb{P} \left(\exists x \in \mathbb{R}^d, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right)$$

$$= \ \mathbb{P} \left[\sup_{x \in C_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta \right] + \mathbb{P} \left[\sup_{x \in C_n^c} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta \right]$$

$$\leq \ \mathbb{P} \left[\sup_{x \in \tilde{U}_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} > \delta \right] + \mathbb{P} \left[\sup_{x \in \tilde{W}_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} > \delta \right]$$

$$+ \mathbb{P} \left[\sup_{x \in C_n^c} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta \right]$$

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$$\leq \quad \mathbb{P}\left[\sup_{x\in \tilde{U}_n\cap U_n(\xi)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta\right] + \mathbb{P}\left[\tilde{U}_n\cap[U_n(\xi)]^c\neq\emptyset\right]$$

$$+\,\mathbb{P}\left[\sup_{x\in W_n(\zeta)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right] + \mathbb{P}\left[\tilde{W}_n\cap[W_n(\zeta)]^c\neq\emptyset\right]$$

$$+\,\mathbb{P}\left[\sup_{x\in C_n^c} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta\right]$$

$$\leq \quad \mathbb{P}\left[\sup_{x\in U_n(\xi)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \ \text{ and } \ \inf_{x\in U_n(\xi)} f_n(x)>0\right] + \mathbb{P}\left[\tilde{U}_n\cap[U_n(\xi)]^c\neq\emptyset\right]$$

$$+\,\mathbb{P}\left[\sup_{x\in W_n(\zeta)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right] + \mathbb{P}\left[\tilde{W}_n\cap[W_n(\zeta)]^c\neq\emptyset\right]$$

$$+\,\mathbb{P}\left[\sup_{x\in C_n^c} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta\right],$$

so that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(\exists x\in\mathbb{R}^d,\ \ f(x)\notin\left[f_n^*(x)-\delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\ ;\ f_n^*(x)+\delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\right]\right)$$

$$\leq \quad \max\left\{\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\sup_{x\in U_n(\xi)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \ \text{ and } \ \inf_{x\in U_n(\xi)} f_n(x)>0\right]\ ;\right.$$

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\tilde{U}_n\cap[U_n(\xi)]^c\neq\emptyset\right]\ ;\ \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\sup_{x\in W_n(\zeta)} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right]\ ;$$

$$\left.\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\tilde{W}_n\cap[W_n(\zeta)]^c\neq\emptyset\right]\ ;\ \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\sup_{x\in C_n^c} \frac{v_n\,|f_n^*(x)-f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta\right]\right\}. \quad (27)$$

- The application of Lemma 1 ensures that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x\in U_n(\xi)} \frac{v_n|f_n^*(x)-f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \ \text{ and } \ \inf_{x\in U_n(\xi)} f_n(x)>0\right] \leq -\frac{\delta^2}{2} \quad (28)$$

and the one of Lemma 2 gives

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x\in W_n(\zeta)} \frac{v_n|f_n^*(x)-f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right] \leq -\delta^2\left(1-\frac{\zeta}{2}\right). \quad (29)$$

- The proof of the following upper bound is quite technical, and is postponed in the appendix (see Section 6.2).

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x\in C_n^c} \frac{v_n|f_n^*(x)-f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} \geq \delta\right] = -\infty. \quad (30)$$

- Since

$$\mathbb{P}\left[\tilde{U}_n\cap[U_n(\xi)]^c\neq\emptyset\right] \quad = \quad \mathbb{P}\left[\exists x_0\in C_n,\ f_n(x_0)\geq\epsilon_n \text{ and } f(x_0)\leq\xi\epsilon_n\right]$$

$$\leq \quad \mathbb{P}\left[\exists x_0\in C_n,\ f_n(x_0)-f(x_0)\geq(1-\xi)\epsilon_n\right]$$

$$\leq \quad \mathbb{P}\left[\frac{1}{\epsilon_n}\sup_{x\in C_n}|f_n(x)-f(x)|\geq(1-\xi)\right],$$

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we get, by application of Theorem 5 in Mokkadem, Pelletier and Worms (2005),

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\tilde{U}_n \cap [U_n(\xi)]^c \neq \emptyset\right]$$
$$\leq \limsup_{n\to\infty} \frac{v_n^2 \epsilon_n^2 h_n^d}{h_n^{*d}} \left\{\frac{1}{nh_n^d \epsilon_n^2} \log \mathbb{P}\left[\frac{1}{\epsilon_n} \sup_{x \in C_n} |f_n(x) - f(x)| \geq (1-\xi)\right]\right\}$$
$$= -\infty. \tag{31}$$

- Similarly, since

$$\mathbb{P}\left[\tilde{W}_n \cap [W_n(\zeta)]^c \neq \emptyset\right] = \mathbb{P}\left[\exists x_0 \in C_n,\ f_n(x_0) < \epsilon_n \text{ and } f(x_0) > \zeta\epsilon_n\right]$$
$$\leq \mathbb{P}\left[\exists x_0 \in C_n,\ f(x_0) - f_n(x_0) > (\zeta-1)\epsilon_n\right]$$
$$\leq \mathbb{P}\left[\frac{1}{\epsilon_n} \sup_{x \in C_n} |f_n(x) - f(x)| > (\zeta-1)\right],$$

the application of Theorem 5 in Mokkadem, Pelletier and Worms (2005) gives

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\tilde{W}_n \cap [W_n(\zeta)]^c \neq \emptyset\right]$$
$$\leq \limsup_{n\to\infty} \frac{v_n^2 \epsilon_n^2 h_n^d}{h_n^{*d}} \left\{\frac{1}{nh_n^d \epsilon_n^2} \log \mathbb{P}\left[\frac{1}{\epsilon_n} \sup_{x \in C_n} |f_n(x) - f(x)| \geq (\zeta-1)\right]\right\}$$
$$= -\infty. \tag{32}$$

The combination of (27)-(32) leads to

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(\exists x \in \mathbb{R}^d,\ \ f(x) \notin \left[f_n^*(x) - \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\ ;\ f_n^*(x) + \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\right]\right) \leq -\delta^2\left(1 - \frac{\zeta}{2}\right).$$

Since this last upper bound holds for any $\zeta > 1$, the upper bound (25) follows.

Proof of the lower bound (26) Set $x_0 \in \cap_n C_n$ such that $f(x_0) \neq 0$, and set $\eta \in]0,1[$. Moreover, let n be large enough so that $f(x_0) \geq \epsilon_n/(1-\eta)$. We then have:

$$v_n |f_n^*(x_0) - f(x_0)| \geq \delta\sqrt{f_n(x_0)\kappa} \ \text{ and } \ f_n(x_0) \geq (1-\eta)f(x_0)$$
$$\Rightarrow \ \frac{v_n |f_n^*(x_0) - f(x_0)|}{\sqrt{f_n(x_0)\kappa}} \geq \delta \ \text{ and } \ f_n(x_0) \geq \epsilon_n$$
$$\Rightarrow \ \frac{v_n |f_n^*(x_0) - f(x_0)|}{\sqrt{f_n(x_0)\kappa}} \geq \delta \ \text{ and } \ x_0 \in \tilde{U}_n$$
$$\Rightarrow \ \sup_{x \in \tilde{U}_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} \geq \delta$$
$$\Rightarrow \ \sup_{x \in C_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} \geq \delta.$$

It follows that

$$\mathbb{P}\left[\sup_{x \in C_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} \geq \delta\right]$$

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$$\geq \quad \mathbb{P}\left[v_n\,|f_n^*(x_0) - f(x_0)| \geq \delta\sqrt{f_n(x_0)\kappa}\right] - \mathbb{P}\left[f_n(x_0) < (1-\eta)f(x_0)\right]$$

$$\geq \quad \mathbb{P}\left[v_n\,|f_n^*(x_0) - f(x_0)| \geq \delta\sqrt{f_n(x_0)\kappa}\right] - \mathbb{P}\left[f(x_0) - f_n(x_0) > \eta f(x_0)\right].$$

Since the application of Corollary 1 in Mokkadem, Pelletier and Worms (2005) ensures that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[f(x_0) - f_n(x_0) > \eta f(x_0)\right] \quad = \quad -\infty,$$

the application of Theorem 1 leads to

$$\liminf_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x\in C_n} \frac{v_n\,|f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} \geq \delta\right] \quad \geq \quad -\frac{\delta^2}{2}. \tag{33}$$

Noting that

$$\mathbb{P}\left(\exists x \in \mathbb{R}^d,\ f(x) \notin \left[f_n^*(x) - \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\ ;\ f_n^*(x) + \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\right]\right)$$

$$\geq \mathbb{P}\left[\sup_{x\in C_n} \frac{v_n\,|f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta\right],$$

the lower bound (26) follows, which concludes the proof of Theorem 4.

5.3.2 Proof of Theorem 3

Set $C_n = C$ for all n, set $\xi \in]0,1[$, $\zeta > 1$, and let $\tilde{U}_n$, $\tilde{W}_n$, $U_n(\xi)$, and $W_n(\zeta)$ be defined according to (20), (21), (22), and (23) respectively.

Upper bound Following the proof of (27), we have:

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(\exists x \in C,\ f(x) \notin \left[f_n^*(x) - \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\ ;\ f_n^*(x) + \delta\frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n}\right]\right)$$

$$\leq \quad \max\left\{\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\sup_{x\in U_n(\xi)} \frac{v_n\,|f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta\ \text{and}\ \inf_{x\in U_n(\xi)} f_n(x) > 0\right]\ ;\right.$$

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\tilde{U}_n \cap [U_n(\xi)]^c \neq \emptyset\right]\ ;\ \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\sup_{x\in W_n(\zeta)} \frac{v_n\,|f_n^*(x) - f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right]\ ;$$

$$\left.\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}}\mathbb{P}\left[\tilde{W}_n \cap [W_n(\zeta)]^c \neq \emptyset\right]\right\}.$$

Moreover, following the proof of (31) and (32), we get

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\tilde{U}_n \cap [U_n(\xi)]^c \neq \emptyset\right] \quad = \quad -\infty,$$

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\tilde{W}_n \cap [W_n(\zeta)]^c \neq \emptyset\right] \quad = \quad -\infty.$$

It thus follows from the application of Lemmas 1 and 2 that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left(\exists x \in C, \ \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right) \leq -\delta^2 \left(1 - \frac{\zeta}{2} \right).$$

Since this last upper bound holds for any $\zeta > 1$, it follows that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left(\exists x \in C, \ \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right) \leq -\frac{\delta^2}{2}.$$

Lower bound Following the proof of (33), we obtain

$$\liminf_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left(\exists x \in C, \ \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{\tilde{T}_n(x)\kappa}}{v_n} \right] \right) \geq -\frac{\delta^2}{2},$$

which concludes the proof of Theorem 3.

5.3.3 Proof of Theorem 2

Upper bound Let (ϵ_n) be a sequence satisfying (13) (the existence of such a sequence is obvious in view of (12)). Moreover, set $C_n = C$ for all n, set $\xi \in]0, 1[$, and let $U_n(\xi)$ be defined according to (22). We note that

$$\mathbb{P} \left(\exists x \in C, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right)$$

$$\leq \mathbb{P} \left[\sup_{x\in C} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} > \delta \ \ \text{and} \ \ \inf_{x\in C} f_n(x) > 0 \right] + \mathbb{P} \left[\inf_{x\in C} f_n(x) = 0 \right]$$

Since, under the assumptions of Theorem 2, there exists $a > 0$ such that $f(x) \geq a$ for all $x \in C$, we have, for n large enough, $U_n(\xi) = C$. It follows that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left(\exists x \in C, \ \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right)$$

$$\leq \max \left\{ \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \mathbb{P} \left[\sup_{x\in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \ \ \text{and} \ \ \inf_{x\in U_n(\xi)} f_n(x) > 0 \right] \ ; \right.$$

$$\left. \limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \mathbb{P} \left[\inf_{x\in C} f_n(x) = 0 \right] \right\}, \tag{34}$$

with, by application of Corollary 2 in Mokkadem, Pelletier and Worms (2005),

$$\lim_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{x\in C} |f_n(x) - f(x)| \geq a \right]$$

$$= \lim_{n\to\infty} \frac{v_n^2 h_n^d}{h_n^{*d}} \left[\frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{x\in C} |f_n(x) - f(x)| \geq a \right] \right]$$

$$= -\infty.$$

The application of Lemma 1 then gives

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left(\exists x \in C, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right) \leq -\frac{\delta^2}{2}.$$

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Lower bound Set $x_0 \in C$; since $f(x_0) \neq 0$, we clearly have, by application of Theorem 1,

$$\liminf_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(\exists x \in C, \ f(x) \notin \left[f_n^*(x) - \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \ ; \ f_n^*(x) + \delta \frac{\sqrt{f_n(x)\kappa}}{v_n} \right] \right)$$

$$\geq \ \liminf_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left(f(x_0) \notin \left[f_n^*(x_0) - \delta \frac{\sqrt{f_n(x_0)\kappa}}{v_n} \ ; \ f_n^*(x_0) + \delta \frac{\sqrt{f_n(x_0)\kappa}}{v_n} \right] \right)$$

$$\geq \ \frac{-\delta^2}{2},$$

which concludes the proof of Theorem 2.

5.4 Proof of Corollaries 1 and 2

Set $H = C$ in the framework of Corollary 1 and $H = \mathbb{R}^d$ in the framework of Corollary 2. Let θ and θ_n satisfy

$$f(\theta) = \sup_{x \in H} f(x) \ \text{ and } \ f_n(\theta_n) = \sup_{x \in H} f_n(x),$$

and let $\tilde{T}_n$ be the truncating function defined as $\tilde{T}_n(x) = \max\left\{ f_n(x); \epsilon_n f(\theta) \right\}$ for all $x \in H$.

Upper bound We first prove that

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n |f_n^*(x) - f(x)| > \delta \sqrt{T_n(x)\kappa} \right] \leq -\frac{\delta^2}{2}. \tag{35}$$

Set $\eta > 0$; since

$$f_n(\theta) > \frac{f(\theta)}{1 + \eta} \ \Rightarrow \ \forall x \in C, \ T_n(x) \geq \frac{\tilde{T}_n(x)}{1 + \eta},$$

we have

$$\mathbb{P}\left[\exists x \in H, \ v_n |f_n^*(x) - f(x)| > \delta \sqrt{T_n(x)\kappa} \right]$$

$$\leq \ \mathbb{P}\left[\exists x \in H, \ v_n |f_n^*(x) - f(x)| > \delta \sqrt{T_n(x)\kappa} \ \text{ and } \ f_n(\theta) > \frac{f(\theta)}{1 + \eta} \right] + \mathbb{P}\left[f_n(\theta) \leq \frac{f(\theta)}{1 + \eta} \right]$$

$$\leq \ \mathbb{P}\left[\exists x \in H, \ v_n |f_n^*(x) - f(x)| > \frac{\delta \sqrt{\tilde{T}_n(x)\kappa}}{\sqrt{1 + \eta}} \right] + \mathbb{P}\left[f(\theta) - f_n(\theta) \geq \frac{\eta f(\theta)}{1 + \eta} \right].$$

The application of Theorem 3 (respectively Theorem 4) in the case $H = C$ (respectively $H = \mathbb{R}^d$) ensures that

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n |f_n^*(x) - f(x)| > \frac{\delta \sqrt{\tilde{T}_n(x)\kappa}}{\sqrt{1 + \eta}} \right] \leq -\frac{\delta^2}{2(1 + \eta)}$$

and the application of Corollary 1 in Mokkadem, Pelletier and Worms (2005) gives

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[f(\theta) - f_n(\theta) \geq \frac{\eta f(\theta)}{1 + \eta} \right]$$

$$\leq \ \limsup_{n \to \infty} \left[\frac{v_n^2 h_n^d}{nh_n^{*d}} \right] \left[\frac{1}{nh_n^d} \log \mathbb{P}\left[f(\theta) - f_n(\theta) \geq \frac{\eta f(\theta)}{1 + \eta} \right] \right]$$

$$= \ -\infty.$$

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Thus, we get

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n|f_n^*(x) - f(x)| > \delta\sqrt{T_n(x)\kappa}\right] \leq -\frac{\delta^2}{2(1+\eta)},$$

and, since η can be chosen arbitrarily close to zero, the proof of the upper bound (35) is completed.

Lower bound We now prove the lower bound

$$\liminf_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n|f_n^*(x) - f(x)| > \delta\sqrt{T_n(x)\kappa}\right] \geq -\frac{\delta^2}{2}. \qquad (36)$$

Set $\eta > 0$; since

$$f_n(\theta_n) \leq (1+\eta)f(\theta) \quad \Rightarrow \quad \forall x \in C, \ T_n(x) \leq (1+\eta)\tilde{T}_n(x),$$

we have

$$\mathbb{P}\left[\exists x \in H, \ v_n\,|f_n^*(x) - f(x)| > \delta\sqrt{T_n(x)\kappa}\right]$$

$$\geq \ \mathbb{P}\left[\exists x \in H, \ v_n\,|f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)\tilde{T}_n(x)\kappa} \ \text{ and } \ f_n(\theta_n) \leq (1+\eta)f(\theta)\right]$$

$$\geq \ \mathbb{P}\left[\exists x \in H, \ v_n\,|f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)\tilde{T}_n(x)\kappa}\right] - \mathbb{P}\left[f_n(\theta_n) - f(\theta) > \eta f(\theta)\right]$$

$$\geq \ \mathbb{P}\left[\exists x \in H, \ v_n\,|f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)\tilde{T}_n(x)\kappa}\right] - \mathbb{P}\left[\sup_{x\in H}|f_n(x) - f(x)| > \eta f(\theta)\right].$$

The application of Theorem 3 (respectively Theorem 4) in the case $H = C$ (respectively $H = \mathbb{R}^d$) ensures that

$$\liminf_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n|f_n^*(x) - f(x)| > \delta\sqrt{(1+\eta)\tilde{T}_n(x)\kappa}\right] \geq -\frac{\delta^2(1+\eta)}{2}$$

and the application of Corollary 2 in Mokkadem, Pelletier and Worms (2005) ensures that

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x\in H}|f_n(x) - f(x)| > \eta f(\theta)\right]$$

$$= \ \limsup_{n\to\infty} \frac{v_n^2 h_n^d}{h_n^{*d}}\left\{\frac{1}{nh_n^d} \log \mathbb{P}\left[\sup_{x\in H}|f_n(x) - f(x)| > \eta f(\theta)\right]\right\}$$

$$= \ -\infty.$$

Thus, it follows that

$$\liminf_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\exists x \in H, \ v_n|f_n^*(x) - f(x)| > \delta\sqrt{T_n(x)\kappa}\right] \geq -\frac{\delta^2}{2},$$

and, since η can be chosen arbitrarily close to zero, the lower bound (36) follows.

6 Appendix

6.1 Proof of Lemmas 1 and 2

The proof of Lemmas 1 and 2 requires the two following preliminary lemmas.

Lemma 3 *Under Assumptions (A1)-(A3), we have*

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \sup_{x\in U_n(\xi)} \log \mathbb{P}\left[\frac{v_n|f_n^*(x)-f(x)|}{\sqrt{f(x)\kappa}} \geq \delta\right] \leq -\frac{\delta^2}{2}.$$

Lemma 4 *Under Assumptions (A1)-(A3), we have*

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \sup_{x\in W_n(\zeta)} \log \mathbb{P}\left[\frac{v_n|f_n^*(x)-f(x)|}{\sqrt{\epsilon_n\kappa}} \geq \delta\right] \leq -\delta^2\left(1-\frac{\zeta}{2}\right).$$

We first prove Lemmas 3 and 4 in Subsection 6.1.1, and then establish Lemmas 1 and 2 in Subsection 6.1.2.

6.1.1 Proof of Lemmas 3 and 4

Set

$$u_n(x) = \begin{cases} \frac{\delta}{\sqrt{f(x)\kappa}} & \text{in the framework of Lemma 3} \\ \frac{\delta}{\sqrt{\epsilon_n\kappa}} & \text{in the framework of Lemma 4} \end{cases}$$

$$\mathcal{E} = \begin{cases} U_n(\xi) & \text{in the framework of Lemma 3} \\ W_n(\zeta) & \text{in the framework of Lemma 4} \end{cases}$$

and, for any $u \in \mathbb{R}$,

$$\Lambda_{n,x}(u) = \frac{v_n^2}{nh_n^{*d}} \log \mathbb{E}\left[\exp\left(\frac{nh_n^{*d}}{v_n}u[f_n^*(x)-f(x)]\right)\right].$$

To study the asymptotics of $\sup_{x\in\mathcal{E}} \mathbb{P}\left[v_n u_n(x)|f_n^*(x)-f(x)| \geq \delta^2\right]$, we first note that, by Chebyshev's inequality, we have

$$\mathbb{P}\left[v_n u_n(x)[f_n^*(x)-f(x)] \geq \delta^2\right]$$
$$= \mathbb{P}\left[\exp\left(\frac{nh_n^{*d}}{v_n}u_n(x)[f_n^*(x)-f(x)]\right) \geq \exp\left(\frac{nh_n^{*d}}{v_n^2}\delta^2\right)\right]$$
$$\leq \exp\left[-\frac{nh_n^{*d}}{v_n^2}\delta^2\right]\mathbb{E}\left(\exp\left[\frac{nh_n^{*d}}{v_n}u_n(x)[f_n^*(x)-f(x)]\right]\right)$$
$$\leq \exp\left[-\frac{nh_n^{*d}}{v_n^2}\delta^2\right]\exp\left[\frac{nh_n^{*d}}{v_n^2}\Lambda_{n,x}(u_n(x))\right]$$

and thus

$$\frac{v_n^2}{nh_n^{*d}} \log \sup_{x\in\mathcal{E}} \mathbb{P}\left[v_n u_n(x)[f_n^*(x)-f(x)] \geq \delta^2\right] \leq -\delta^2 + \sup_{x\in\mathcal{E}} \Lambda_{n,x}(u_n(x)). \tag{37}$$

In the same way, we prove that

$$\frac{v_n^2}{nh_n^{*d}} \log \sup_{x \in \mathcal{E}} \mathbb{P}\left[\, v_n u_n(x)[f(x) - f_n^*(x)] \geq \delta^2 \,\right] \;\leq\; -\delta^2 + \sup_{x \in \mathcal{E}} \Lambda_{n,x}(-u_n(x)). \tag{38}$$

Let us set $e \in \{-1, +1\}$ and let us at first assume that

$$\Lambda_{n,x}(eu_n(x)) = \frac{u_n^2(x)}{2}\kappa f(x) + R_{n,x}(eu_n(x)) \quad \text{with} \quad \lim_{n \to \infty} \sup_{x \in \mathcal{E}} R_{n,x}(eu_n(x)) = 0. \tag{39}$$

• In the framework of Lemma 3, (39) means that

$$\Lambda_{n,x}(eu_n(x)) = \frac{\delta^2}{2} + R_{n,x}(eu_n(x)) \quad \text{with} \quad \lim_{n \to \infty} \sup_{x \in U_n(\xi)} R_{n,x}(eu_n(x)) = 0,$$

and Lemma 3 is thus a straightforward consequence of (37) and (38).
• In the framework of Lemma 4, (39) can be rewritten as

$$\Lambda_{n,x}(eu_n(x)) = \frac{\delta^2}{2\epsilon_n}f(x) + R_{n,x}(eu_n(x)) \quad \text{with} \quad \lim_{n \to \infty} \sup_{x \in W_n(\zeta)} R_{n,x}(eu_n(x)) = 0,$$

and, since $\sup_{x \in W_n(\zeta)} f(x)/\epsilon_n \leq \zeta$, Lemma 4 is also given by (37) and (38).

It remains to prove (39). Let us first note that

$$\begin{aligned}
\Lambda_{n,x}(eu_n(x)) &= -v_n eu_n(x)f(x) + \frac{v_n^2}{nh_n^{*d}} \log \mathbb{E}\left[\exp\left(v_n^{-1}eu_n(x)\sum_{i=1}^{n}K\left(\frac{x - X_i}{h_n^*}\right)\right)\right] \\
&= -v_n eu_n(x)f(x) + \frac{v_n^2}{h_n^{*d}} \log \mathbb{E}\left[\exp\left\{v_n^{-1}eu_n(x)K\left(\frac{x - X_1}{h_n^*}\right)\right\}\right]
\end{aligned}$$

By Taylor formula for the function $\log$, there exists c_n between 1 and $\mathbb{E}\left[\exp\left\{v_n^{-1}eu_n(x)K\left(\frac{x-X_1}{h_n^*}\right)\right\}\right]$ such that

$$\Lambda_{n,x}(eu_n(x)) = -v_n eu_n(x)f(x) + \frac{v_n^2}{h_n^{*d}}\mathbb{E}\left[\exp\left\{v_n^{-1}eu_n(x)K\left(\frac{x - X_1}{h_n^*}\right)\right\} - 1\right] - R_{n,x}^{(1)}(eu_n(x))$$

with

$$R_{n,x}^{(1)}(eu_n(x)) = \frac{v_n^2}{2c_n^2 h_n^{*d}}\left\{\mathbb{E}\left[\exp\left\{v_n^{-1}eu_n(x)K\left(\frac{x - X_1}{h_n^*}\right)\right\} - 1\right]\right\}^2. \tag{40}$$

We rewrite $\Lambda_{n,x}(eu_n(x))$ as

$$\begin{aligned}
&\Lambda_{n,x}(eu_n(x)) \\
&= -v_n eu_n(x)f(x) + \frac{v_n^2}{h_n^{*d}}\int_{\mathbb{R}^d}\left\{\exp\left[v_n^{-1}eu_n(x)K\left(\frac{x - y}{h_n^*}\right)\right] - 1\right\}f(y)\,dy - R_{n,x}^{(1)}(eu_n(x)) \\
&= -v_n eu_n(x)f(x) + v_n^2\int_{\mathbb{R}^d}\left(\exp\left[v_n^{-1}eu_n(x)K(z)\right] - 1\right)f(x - h_n^*z)\,dz - R_{n,x}^{(1)}(eu_n(x)) \\
&= -v_n eu_n(x)f(x) + v_n^2\int_{\mathbb{R}^d}\left(v_n^{-1}eu_n(x)K(z) + \frac{v_n^{-2}u_n^2(x)K^2(z)}{2}\right)f(x - h_n^*z)\,dz \\
&\quad - R_{n,x}^{(1)}(eu_n(x)) + R_{n,x}^{(2)}(eu_n(x))
\end{aligned}$$

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with

$$R_{n,x}^{(2)}(eu_n(x)) \leq v_n^2 \int_{\mathbb{R}^d} \frac{v_n^{-3} u_n^3(x)|K(z)|^3}{6} \exp\left[v_n^{-1} u_n(x)|K(z)|\right] f(x - h_n^* z)\, dz$$

$$\leq \frac{v_n^{-1} u_n^3(x)}{6} \|f\|_\infty \exp\left[v_n^{-1} u_n(x)\|K\|_\infty\right] \int_{\mathbb{R}^d} |K(z)|^3\, dz. \tag{41}$$

It follows that

$$\Lambda_{n,x}(eu_n(x))$$
$$= -v_n eu_n(x) \int_{\mathbb{R}^d} K(z) \left[f(x) - f(x - h_n^* z)\right] dz + \frac{u_n^2(x)}{2} \int_{\mathbb{R}^d} K^2(z) f(x - h_n^* z)\, dz$$
$$- R_{n,x}^{(1)}(eu_n(x)) + R_{n,x}^{(2)}(eu_n(x))$$

and, setting

$$R_{n,x}^{(3)}(eu_n(x)) = v_n eu_n(x) \int_{\mathbb{R}^d} K(z) \left[f(x) - f(x - h_n^* z)\right] dz, \tag{42}$$

we obtain

$$\Lambda_{n,x}(eu_n(x))$$
$$= \frac{u_n^2(x)}{2} \int_{\mathbb{R}^d} K^2(z) \left[f(x - h_n^* z) - f(x)\right] dz + \frac{u_n^2(x) f(x) \kappa}{2} - R_{n,x}^{(1)}(eu_n(x))$$
$$+ R_{n,x}^{(2)}(eu_n(x)) - R_{n,x}^{(3)}(eu_n(x))$$
$$= \frac{u_n^2(x) f(x) \kappa}{2} - R_{n,x}^{(1)}(eu_n(x)) + R_{n,x}^{(2)}(eu_n(x)) - R_{n,x}^{(3)}(eu_n(x)) + R_{n,x}^{(4)}(eu_n(x))$$

where

$$R_{n,x}^{(4)}(eu_n(x)) = \frac{u_n^2(x)}{2} \int_{\mathbb{R}^d} K^2(z) \left[f(x - h_n^* z) - f(x)\right] dz. \tag{43}$$

To conclude the proof of Lemmas 3 and 4, it remains to show that

$$\lim_{n \to \infty} \sup_{x \in \mathcal{E}} R_{n,x}^{(i)}(eu_n(x)) = 0 \quad \text{for} \quad i \in \{1, \ldots, 4\}.$$

Let c and c' denote generic positive constants that may vary from line to line. We shall use several times the fact that

$$\sup_{x \in \mathcal{E}} u_n(x) \leq \frac{c}{\sqrt{\epsilon_n}}.$$

• To study $R_{n,x}^{(1)}(eu_n(x))$ defined in (40), we first note that

$$\exp\left\{v_n^{-1} eu_n(x) K\left(\frac{x - X_1}{h_n^*}\right)\right\} \geq \exp\left\{-v_n^{-1} u_n(x)\|K\|_\infty\right\}$$

so that

$$\frac{1}{c_n^2} \leq \exp\left\{2 v_n^{-1} u_n(x)\|K\|_\infty\right\}.$$

Moreover, since

$$\left| \mathbb{E}\left[\exp\left\{v_n^{-1} eu_n(x) K\left(\frac{x - X_1}{h_n^*}\right)\right\} - 1\right] \right|$$
$$\leq \mathbb{E}\left[\left|v_n^{-1} eu_n(x) K\left(\frac{x - X_1}{h_n^*}\right)\right| \exp\left|v_n^{-1} eu_n(x) K\left(\frac{x - X_1}{h_n^*}\right)\right|\right]$$
$$\leq h_n^{*d} v_n^{-1} u_n(x) \|f\|_\infty \exp\left\{v_n^{-1} u_n(x)\|K\|_\infty\right\} \int_{\mathbb{R}^d} |K(z)|\, dz,$$

we obtain

$$
\begin{aligned}
\sup_{x \in \mathcal{E}} |R_{n,x}^{(1)}(eu_n(x))| \quad &\leq \quad \sup_{x \in \mathcal{E}} h_n^{*d} \frac{u_n^2(x)}{2} \exp\left\{4v_n^{-1} u_n(x)\|K\|_\infty\right\} \|f\|_\infty^2 \left(\int_{\mathbb{R}^d} |K(z)|\, dz\right)^2 \\
&\leq \quad c \frac{h_n^{*d}}{\epsilon_n} \exp\left(\frac{c'}{v_n \sqrt{\epsilon_n}}\right) \\
&\rightarrow \quad 0 \quad \text{since} \quad h_n^{*d}/\epsilon_n \rightarrow 0 \quad \text{and} \quad v_n \sqrt{\epsilon_n} \rightarrow \infty.
\end{aligned}
$$

- It follows from (41) that

$$
\begin{aligned}
\sup_{x \in \mathcal{E}} |R_{n,x}^{(2)}(eu_n(x))| \quad &\leq \quad \frac{c}{v_n \epsilon_n^{3/2}} \exp\left(\frac{c}{v_n \sqrt{\epsilon_n}}\right) \\
&\rightarrow \quad 0 \quad \text{since} \quad v_n \epsilon_n^{3/2} \rightarrow \infty.
\end{aligned}
$$

- To upper bound $R_{n,x}^{(3)}$ defined in (42), we use a Taylor expansion and Assumptions (A1)-(A3) to obtain

$$
\sup_{x \in \mathcal{E}} \left| \int_{\mathbb{R}^d} K(z) \left[f(x) - f(x - h_n^* z)\right]\, dz \right| \leq h_n^{*2} \sup_{x \in \mathbb{R}^d} \|D^2 f(x)\| \int_{\mathbb{R}^d} \|z\|^2 K(z) dz,
$$

from which we deduce that

$$
\begin{aligned}
\sup_{x \in \mathcal{E}} |R_{n,x}^{(3)}(eu_n(x))| \quad &\leq \quad c \frac{v_n h_n^{*2}}{\sqrt{\epsilon_n}} \\
&\rightarrow \quad 0 \quad \text{since} \quad \frac{v_n h_n^{*2}}{\sqrt{\epsilon_n}} \rightarrow 0.
\end{aligned}
$$

- Similarly, for $R_{n,x}^{(4)}$ defined in (43), we note that

$$
\sup_{x \in \mathcal{E}} \left| \int_{\mathbb{R}^d} K^2(z) \left[f(x) - f(x - h_n^* z)\right]\, dz \right| \leq h_n^* \|K\|_\infty \sup_{x \in \mathbb{R}^d} \|\nabla f(x)\| \int_{\mathbb{R}^d} \|z\| K(z) dz,
$$

so that

$$
\begin{aligned}
\sup_{x \in \mathcal{E}} |R_{n,x}^{(4)}(eu_n(x))| \quad &\leq \quad c \frac{h_n^*}{\epsilon_n} \\
&\rightarrow \quad 0 \quad \text{since} \quad \frac{h_n^*}{\epsilon_n} \rightarrow 0,
\end{aligned}
$$

which concludes the proof of Lemmas 3 and 4.

6.1.2 Proof of Lemmas 1 and 2

In view of Assumption (A2), there exists β and $\|K\|_H$ such that

$$
|K(x) - K(y)| \leq \|K\|_H \|x - y\|^\beta \quad \forall x, y \in \mathbb{R}^d.
$$

Set $b = \sup_{x \in \mathbb{R}^d} \|\nabla f(x)\|$, $\delta > 0$, $\rho \in]0, \delta[$, and

$$
R_n \quad = \quad \left(\frac{\rho h_n^{*(d+\beta)} \sqrt{\xi \epsilon_n \kappa}}{2 [2\sqrt{d}]^\beta v_n (b + \|K\|_H)}\right)^{1/\beta}.
$$

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Let m be the integer satisfying $w_n/R_n \leq m < 1 + w_n/R_n$, and set $N'(n) = m^d$. The ball $D_n = \left\{ x \in \mathbb{R}^d, \ \|x\| \leq w_n \right\}$ can be covered by $N'(n)$ cubes with lenght side $2R_n$ (note that $U_n(\xi)$, $W_n(\zeta)$, and C_n are subsets of D_n). We denote by $B_n^{(i)}$, $i = 1, \ldots, N(n)$ ($N(n) \leq N'(n)$), the cubes that intersect $U_n(\xi)$, and by $\tilde{B}_n^{(i)}$, $i = 1, \ldots, \tilde{N}(n)$ ($\tilde{N}(n) \leq N'(n)$), the cubes that intersect $W_n(\zeta)$. Moreover, for each $i \in \{1, \ldots, N(n)\}$, we choose $x_n^{(i)} \in B_n^{(i)} \cap U_n(\xi)$, and for each $i \in \{1, \ldots, \tilde{N}(n)\}$, we choose $\tilde{x}_n^{(i)} \in \tilde{B}_n^{(i)} \cap W_n(\zeta)$.

To prove Lemma 1, we first note that

$$
\mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n \, |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta \right] \quad \leq \quad \sum_{i=1}^{N(n)} \mathbb{P}\left[\sup_{x \in B_i^{(n)} \cap U_n(\xi)} \frac{v_n \, |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta \right]
$$

$$
\leq \quad N(n) \max_{1 \leq i \leq N(n)} \mathbb{P}\left[\sup_{x \in B_i^{(n)} \cap U_n(\xi)} \frac{v_n \, |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta \right].
$$

Now, for any $i \in \{1, \ldots, N(n)\}$ and for any $x \in B_i^{(n)} \cap U_n(\xi)$, we write

$$
\frac{v_n \, |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \quad \leq \quad \frac{v_n \left| f_n^*(x) - f_n^*(x_i^{(n)}) \right|}{\sqrt{f(x)\kappa}} + \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x)\kappa}} + \frac{v_n \left| f(x_i^{(n)}) - f(x) \right|}{\sqrt{f(x)\kappa}}.
$$

On the one hand, since K is Hölder continuous and since $x \in B_i^{(n)} \Rightarrow \|x - x_i^{(n)}\| \leq 2\sqrt{d}R_n$, we get

$$
\frac{v_n \left| f_n^*(x) - f_n^*(x_i^{(n)}) \right|}{\sqrt{f(x)\kappa}} \quad \leq \quad \frac{v_n \|K\|_H}{\sqrt{\xi\epsilon_n\kappa} \ h_n^{*d}} \left\| \frac{x - x_i^{(n)}}{h_n^*} \right\|^\beta
$$

$$
\leq \quad \frac{v_n \|K\|_H [2\sqrt{d}R_n]^\beta}{h_n^{*(d+\beta)}\sqrt{\xi\epsilon_n\kappa}}
$$

$$
\leq \quad \frac{\rho}{2}.
$$

On the other hand, we have

$$
\frac{v_n \left| f(x_i^{(n)}) - f(x) \right|}{\sqrt{f(x)\kappa}} \quad \leq \quad \frac{v_n b \left\| x - x_i^{(n)} \right\|}{\sqrt{\xi\epsilon_n\kappa}},
$$

and, since $\beta \leq 1$ and $R_n \to 0$, we obtain, for n large enough,

$$
\frac{v_n \left| f(x_i^{(n)}) - f(x) \right|}{\sqrt{f(x)\kappa}} \quad \leq \quad \frac{v_n b \left\| x - x_i^{(n)} \right\|}{\sqrt{\xi\epsilon_n\kappa}}
$$

$$
\leq \quad \frac{v_n b R_n^\beta}{\sqrt{\xi\epsilon_n\kappa}}
$$

$$
\leq \quad \frac{\rho}{2}.
$$

We deduce that, for all n sufficiently large, $\forall i \in \{1, \ldots, N(n)\}$, $\forall x \in B_i^{(n)} \cap U_n(\xi)$,

$$
\frac{v_n \, |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \quad \leq \quad \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x)\kappa}} + \rho
$$

$$\leq \sqrt{1 + \frac{f(x_i^{(n)}) - f(x)}{f(x)}} \; \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x_i^{(n)})\kappa}} + \rho$$

$$\leq \sqrt{1 + \frac{bR_n}{f(x)}} \; \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x_i^{(n)})\kappa}} + \rho$$

$$\leq \sqrt{1 + \frac{bR_n}{\xi\epsilon_n}} \; \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x_i^{(n)})\kappa}} + \rho$$

$$\leq \sqrt{1 + \rho} \; \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x_i^{(n)})\kappa}} + \rho.$$

We can then deduce that, for n large enough,

$$\mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta\right] \leq N(n) \max_{1 \leq i \leq N(n)} \mathbb{P}\left[\frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{f(x_i^{(n)})\kappa}} \geq \frac{\delta - \rho}{\sqrt{1 + \rho}}\right]$$

$$\leq N(n) \sup_{x \in U_n(\xi)} \mathbb{P}\left[\frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \frac{\delta - \rho}{\sqrt{1 + \rho}}\right].$$

Applying Lemma 3, we obtain

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta\right]$$

$$\leq \limsup_{n \to \infty} \left\{\frac{v_n^2 \log N(n)}{nh_n^{*d}}\right\} - \frac{(\delta - \rho)^2}{2(1 + \rho)}$$

$$\leq \limsup_{n \to \infty} \left\{\frac{dv_n^2}{\beta nh_n^{*d}}\left[\beta \log(w_n) - (d + \beta)\log h_n^* - \frac{\log \epsilon_n}{2} + \log v_n\right]\right\} - \frac{(\delta - \rho)^2}{2(1 + \rho)}$$

$$\leq -\frac{(\delta - \rho)^2}{2(1 + \rho)} \quad \left(\text{since} \quad \frac{v_n^2 \log h_n^*}{nh_n^{*d}} \to 0, \quad \frac{v_n^2 \log v_n}{nh_n^{*d}} \to 0, \quad \text{and} \quad \frac{v_n^2 \log \epsilon_n}{nh_n^{*d}} \to 0\right).$$

This last upper bound holding for any $\rho \in]0, \delta[$, it follows that

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \delta\right] \leq -\frac{\delta^2}{2}. \tag{44}$$

To conclude the proof of Lemma 1, let us now set $\eta > 0$, and note that

$$\mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \quad \text{and} \quad \inf_{x \in U_n(\xi)} f_n(x) > 0\right]$$

$$\leq \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{f(x)}{f_n(x)} \geq 1 + \eta \quad \text{and} \quad \inf_{x \in U_n(\xi)} f_n(x) > 0\right]$$

$$+ \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \frac{\delta}{\sqrt{1 + \eta}}\right]$$

$$\leq \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{f(x) - f_n(x)}{f(x)} \geq \frac{\eta}{1 + \eta}\right] + \mathbb{P}\left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f(x)\kappa}} \geq \frac{\delta}{\sqrt{1 + \eta}}\right].$$

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Since $x \in U_n(\xi) \Rightarrow f(x) \geq \xi \epsilon_n$ and since $v_n^2 h_n^d \epsilon_n^2 / h_n^{*d} \to \infty$, we have, by application of Theorem 5 in Mokkadem, Pelletier and Worms (2005),

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \mathbb{P} \left[\sup_{x \in U_n(\xi)} \frac{f(x) - f_n(x)}{f(x)} \geq \frac{\eta}{1+\eta} \right]$$

$$\leq \limsup_{n \to \infty} \frac{v_n^2 h_n^d \epsilon_n^2}{h_n^{*d}} \left\{ \frac{1}{nh_n^d \epsilon_n^2} \mathbb{P} \left[\frac{1}{\epsilon_n} \sup_{x \in C_n} |f_n(x) - f(x)| \geq \frac{\xi \eta}{1+\eta} \right] \right\}$$

$$= -\infty.$$

Now, in view of (44), we deduce that

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{x \in U_n(\xi)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{f_n(x)\kappa}} \geq \delta \right] \leq \frac{-\delta^2}{2(1+\eta)},$$

and, since this last upper bound holds for all $\eta > 0$, Lemma 1 follows.

To prove Lemma 2, we proceed exactly in the same way as for establihing (44); we first note that

$$\mathbb{P} \left[\sup_{x \in W_n(\zeta)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right] \leq \tilde{N}(n) \max_{1 \leq i \leq \tilde{N}(n)} \mathbb{P} \left[\sup_{x \in \tilde{B}_i^{(n)} \cap W_n(\zeta)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right].$$

with, for any $i \in \{1, \dots, \tilde{N}(n)\}$, $x \in \tilde{B}_i^{(n)} \cap W_n(\zeta)$, and n large enough,

$$\frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \leq \frac{v_n \left| f_n^*(x) - f_n^*(\tilde{x}_i^{(n)}) \right|}{\sqrt{\epsilon_n \kappa}} + \frac{v_n \left| f_n^*(\tilde{x}_i^{(n)}) - f(\tilde{x}_i^{(n)}) \right|}{\sqrt{\epsilon_n \kappa}} + \frac{v_n \left| f(\tilde{x}_i^{(n)}) - f(x) \right|}{\sqrt{\epsilon_n \kappa}}$$

$$\leq \frac{v_n \left| f_n^*(x_i^{(n)}) - f(x_i^{(n)}) \right|}{\sqrt{\epsilon_n \kappa}} + \rho$$

We then deduce that

$$\mathbb{P} \left[\sup_{x \in W_n(\zeta)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right] \leq \tilde{N}(n) \sup_{x \in W_n(\zeta)} \mathbb{P} \left[\frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta - \rho \right]$$

and, applying Lemma 4, we obtain

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{x \in W_n(\zeta)} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right]$$

$$\leq \limsup_{n \to \infty} \frac{v_n^2 \log \tilde{N}(n)}{nh_n^{*d}} - (\delta - \rho)^2 \left(1 - \frac{\zeta}{2} \right)$$

$$\leq -(\delta - \rho)^2 \left(1 - \frac{\zeta}{2} \right).$$

Since this last upper bound holds for any $\rho > 0$, Lemma 2 follows.

6.2 Proof of (30)

Since $\tilde{T}_n(x) \geq \epsilon_n$ for all $x \in \mathbb{R}^d$, we have

$$\limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{x \in C_n^c} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\tilde{T}_n(x)\kappa}} > \delta \right]$$

$$\leq \limsup_{n \to \infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{\|x\| > w_n} \frac{v_n |f_n^*(x) - f(x)|}{\sqrt{\epsilon_n \kappa}} > \delta \right].$$

Now, recall that $\sup_{\|x\|\in\mathbb{R}^d}|\mathbb{E}(f_n^*(x))-f(x)|=O(h_n^{*2})$; in view of the condition $v_n h_n^{*2}/\sqrt{\epsilon_n}\to 0$ in (13), Equation (30) is thus a straightforward consequence of the asymptotic

$$\limsup_{n\to\infty}\frac{v_n^2}{nh_n^{*d}}\log\mathbb{P}\left[\sup_{\|x\|\geq w_n}\frac{v_n|f_n^*(x)-\mathbb{E}(f_n^*(x))|}{\sqrt{\epsilon_n\kappa}}\geq\delta\right]=-\infty. \tag{45}$$

To prove (45), we need the following technical Lemma.

Lemma 5 *Assume that (A1) and (A4) hold. For all $\gamma>0$, we have*

$$\sup_{\|x\|\geq w_n}\int_{\mathbb{R}^d}K^2\left(\frac{x-y}{h_n^*}\right)f(y)dy\leq\gamma h_n^{*d}\epsilon_n.$$

Proof of Lemma 5 Set $\gamma>0$, and write

$$\frac{1}{h_n^{*d}\epsilon_n}\int_{\mathbb{R}^d}K^2\left(\frac{x-y}{h_n^*}\right)f(y)dy$$
$$=\frac{1}{\epsilon_n}\int_{\|z\|\leq w_n/2}K^2(z)f(x-h_n^*z)dz+\frac{1}{\epsilon_n}\int_{\|z\|>w_n/2}K^2(z)f(x-h_n^*z)dz$$

On the one hand, we note that

$$\|x\|\geq w_n\text{ and }\|z\|\leq w_n/2\Rightarrow\|x-h_n^*z\|\geq w_n|1-h_n^*/2|$$
$$\Rightarrow\|x-h_n^*z\|\geq w_n/2\text{ for }n\text{ large enough}$$
$$\Rightarrow f(x-h_n^*z)\leq M_f2^q w_n^{-q}\text{ for }n\text{ large enough}$$

(where $M_f=\sup_{x\in\mathbb{R}^d}\|x\|^q f(x)$), so that

$$\sup_{\|x\|\geq w_n}\frac{1}{\epsilon_n}\int_{\|z\|\leq w_n/2}K^2(z)f(x-h_n^*z)dz\leq\frac{M_f2^q}{w_n^q\epsilon_n}\int_{\mathbb{R}^d}K^2(z)dz$$
$$\leq\frac{\gamma}{2}\text{ for }n\text{ large enough}$$

(since $w_n^q\epsilon_n\to\infty$).
On the other hand, we have

$$\sup_{\|x\|>w_n}\frac{1}{\epsilon_n}\int_{\|z\|>w_n/2}K^2(z)f(x-h_n^*z)dz\leq\sup_{\|x\|>w_n}\frac{\|f\|_\infty 2^q}{w_n^q\epsilon_n}\int_{\mathbb{R}^d}\|z\|^q K^2(z)dz$$
$$\leq\frac{\gamma}{2}\text{ for }n\text{ large enough}$$

which concludes the proof of Lemma 5. □

Let us now come back to the proof of (45). Set

$$\mathcal{F}_n=\left\{K\left(\frac{x-\cdot}{h_n^*}\right);\ \|x\|>w_n\right\}.$$

The classes $\mathcal{F}_n$, $n\geq 1$, are contained in the class $\mathcal{F}(K)$ defined by (9); since K satisfies the covering number condition (10), there exist $A>0$ and $v>0$ such that, $\forall\epsilon>0$, $\forall n\geq 1$,

$$N_2(\epsilon\|K\|_\infty,\mathbb{P},\mathcal{F}_n)\leq\left(\frac{A}{\epsilon}\right)^v.$$

Now, let us take $U = \|K\|_\infty$ and $\sigma^2 = \gamma h_n^{*d} \epsilon_n$ with $\gamma > 0$. Since $h_n^{*d} \epsilon_n \to 0$ and in view of Lemma 5, we have, for n sufficiently large,

$$\sigma \leq U/2$$

and

$$\sigma^2 \geq \mathbb{E}\left[K^2\left(\frac{x - X_1}{h_n^*}\right)\right] \quad \text{for} \quad \|x\| \geq w_n.$$

Thus, it follows from Theorem 2.1 of Giné and Guillou (2002) that there exist two constants M and L depending only on A and v such that, for

$$t \geq L\left(U \log \frac{U}{\sigma} + \sqrt{n}\sigma\sqrt{\log \frac{U}{\sigma}}\right),$$

we have

$$\log \mathbb{P}\left[\sup_{\|x\| \geq w_n} \left|\sum_{i=1}^n \left[K\left(\frac{x - X_i}{h_n^*}\right) - \mathbb{E}\left(K\left(\frac{x - X_i}{h_n^*}\right)\right)\right]\right| \geq t\right]$$
$$\leq \log M - \frac{t}{MU} \log\left(1 + \frac{tU}{M[\sqrt{n}\sigma + U\sqrt{\log(U/\sigma)}]^2}\right). \tag{46}$$

It follows from the conditions $h_n^*/\epsilon_n \to 0$, $v_n \epsilon_n^{3/2} \to \infty$ and $nh_n^{*d}/[v_n^2 \log(1/h_n^*)] \to \infty$ in (13) that $\lim_{n \to \infty} \sqrt{n}\sigma/[U\sqrt{\log \frac{U}{\sigma}}] = \infty$, and thus, for n large enough,

$$\sqrt{n}\sigma + U\sqrt{\log \frac{U}{\sigma}} \leq 2\sqrt{n}\sigma. \tag{47}$$

Now, set $t_n = \delta n h_n^{*d}\sqrt{\epsilon_n \kappa}/v_n$. It follows from (47) that, for n large enough,

$$\frac{U \log \frac{U}{\sigma} + \sqrt{n}\sigma\sqrt{\log \frac{U}{\sigma}}}{t_n} \leq \frac{2\sqrt{n}\sigma\sqrt{\log \frac{U}{\sigma}}}{t_n}$$
$$\leq \sqrt{\frac{4v_n^2 \log(U/[\gamma\epsilon_n h_n^{*d}])}{\kappa n h_n^{*d}}}.$$

The conditions $h_n^*/\epsilon_n \to 0$ and $nh_n^{*d}/[v_n^2 \log(1/h_n^*)] \to \infty$ in (13) ensure then that, for n large enough,

$$t_n \geq L\left(U \log \frac{U}{\sigma} + \sqrt{n}\sigma\sqrt{\log \frac{U}{\sigma}}\right). \tag{48}$$

Since

$$\log \mathbb{P}\left[\sup_{\|x\| \geq w_n} \frac{v_n|f_n^*(x) - \mathbb{E}(f_n^*(x))|}{\sqrt{\epsilon_n \kappa}} \geq \delta\right]$$
$$= \log \mathbb{P}\left[\sup_{\|x\| \geq w_n} \left|\sum_{i=1}^n \left[K\left(\frac{x - X_i}{h_n^*}\right) - \mathbb{E}\left(K\left(\frac{x - X_i}{h_n^*}\right)\right)\right]\right| \geq t_n\right],$$

it follows from (46), (47) and (48) that, for n large enough,

$$\log \mathbb{P}\left[\sup_{\|x\| \geq w_n} \frac{v_n|f_n^*(x) - \mathbb{E}(f_n^*(x))|}{\sqrt{\epsilon_n \kappa}} \geq \delta\right] \leq \log M - \frac{t_n}{MU} \log\left(1 + \frac{t_n U}{4Mn\sigma^2}\right).$$

Noting that

$$\lim_{n\to\infty} \frac{t_n U}{4Mn\sigma^2} = \lim_{n\to\infty} \frac{\delta U}{4M\gamma v_n \sqrt{\epsilon_n}} = 0,$$

we obtain

$$\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \log \mathbb{P} \left[\sup_{\|x\|\geq w_n} \frac{v_n|f_n^*(x) - \mathbb{E}(f_n^*(x))|}{\sqrt{\epsilon_n \kappa}} \geq \delta \right] \leq -\limsup_{n\to\infty} \frac{v_n^2}{nh_n^{*d}} \frac{t_n^2}{4M^2 n\sigma^2}$$

$$\leq -\frac{\delta^2 \kappa}{4\gamma M^2},$$

which implies (45) by letting $\gamma \to 0$.

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